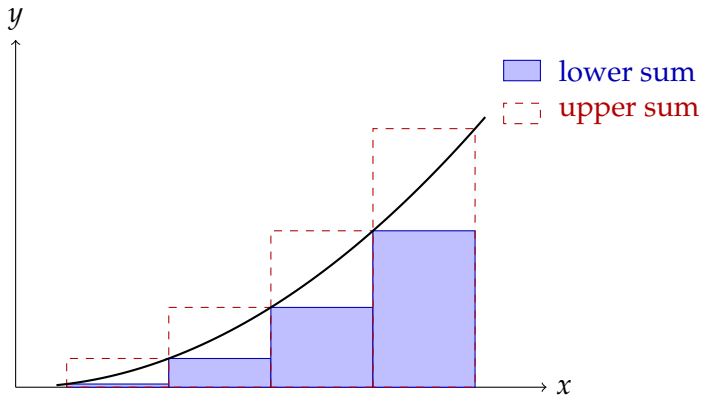
Riemann Sums in Disguise

David Liu



The integral is trapped between lower and upper sums.

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The Technique in Two Sentences

When to reach for it

A limit of $\frac{1}{n} \cdot (\text{a sum})$, an integral of something with a floor in it, or a sum over an arbitrary partition $x_0 < x_1 < \cdots < x_n$ whose terms look like (width)$\times$(height).

The move

Rewrite the expression as $\sum f(x_i)\,\Delta x_i$ and recognise $\int f$. If the partition is arbitrary, compare the sum with the integral using convexity.

Floors Become Sums

On each unit interval $[k, k+1)$ the floor is constant — a floor integral is really a sum over k; read off the Riemann-sum shape and drop the rest.

Floors Become Sums: The Move

When a floor integral or a partition sum shows up, rewrite it as a Riemann sum and read off the limit.

Floor constant on each unit interval

On $[k, k+1)$, $\lfloor x \rfloor = k$ is constant, so

$$\int_0^n (\text{floor stuff})\, dx = \sum_{k=0}^{n-1} \int_k^{k+1} (\text{constant in } x)\, dx.$$

Then read off the shape

After the sum lands on a $\frac{1}{n}\sum f(k/n)$ pattern, lower-order terms like $O(1/n)$ vanish in the limit — drop them and the limit is the integral.

Trapezoids vs. Integrals

For convex f the chord sits above the graph — so a trapezoidal sum overestimates the integral, and refining the mesh drives it down to the integral.

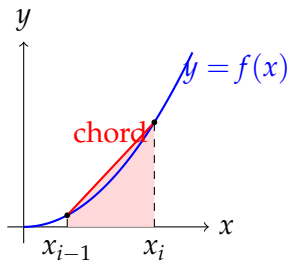
Trapezoid Rule and Convex Functions

For convex f the secant line sits above the graph — so the trapezoid overestimates the integral, and converges to it as the mesh refines.

A sum of the form

$$\sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} (x_i - x_{i-1})$$

is a *trapezoidal Riemann sum*: each term is the area of the trapezoid under the chord joining two points of the graph.



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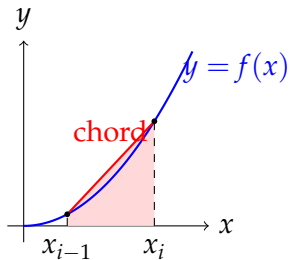

For convex f the chord always lies above the graph, so every trapezoid overestimates its piece of the integral: **each trapezoidal sum is** $\geq \int f$, and the infimum over all partitions is the integral itself.

Example: Floors Become Sums

The floor turns the integral into a sum, and the sum was a Riemann sum all along.

Example: Floors Become Sums

For a real number n, let $\lfloor n \rfloor$ be the greatest integer less than or equal to n. Compute

$$\lim_{n \to \infty} \frac{1}{n^3} \int_0^n x \lfloor x \rfloor \, dx.$$

1

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Solution: Floors Become Sums

The floor turned the integral into a sum, and the sum was a Riemann sum all along.

Why we may take n an integer: for real n the leftover piece $\int_{\lfloor n \rfloor}^{n} x\lfloor x \rfloor \, dx \leq n^2$ is tiny next to n^3.

2

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Why split at the integers: on $[k, k+1)$, $\lfloor x \rfloor = k$ is constant, so it comes out of the integral.

$$\int_0^n x\lfloor x \rfloor \, dx = \sum_{k=0}^{n-1} k \int_k^{k+1} x \, dx$$

2

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Why split at the integers: on $[k, k+1)$, $\lfloor x \rfloor = k$ is constant, so it comes out of the integral.

$$\begin{aligned}\int_0^n x\lfloor x \rfloor \, dx &= \sum_{k=0}^{n-1} k \int_k^{k+1} x \, dx \\ &= \sum_{k=0}^{n-1} k \cdot \frac{(k+1)^2 - k^2}{2} = \frac{1}{2} \sum_{k=0}^{n-1} k(2k+1).\end{aligned}$$

2

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Solution: Floors Become Sums (cont.)

Why split $k(2k+1) = 2k^2 + k$*:* a Riemann sum needs the shape $\frac{1}{n}\sum f\left(\frac{k}{n}\right)$, so we make $\frac{k}{n}$ appear and set aside what does not fit.

$$\frac{1}{n^3} \cdot \frac{1}{2} \sum_{k=0}^{n-1} k(2k+1) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^2 + \frac{1}{2n^3} \sum_{k=0}^{n-1} k$$

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Why the leftover dies: $\sum_{k<n} k < n^2$, so the second term is below $\frac{1}{2n}$.

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Why the leftover dies: $\sum_{k<n} k < n^2$, so the second term is below $\frac{1}{2n}$. The surviving sum is exactly a Riemann sum:

$$\frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^2 \longrightarrow \int_0^1 x^2 \, dx = \boxed{\frac{1}{3}}.$$

3

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Example: The Trapezoid Rule

Factor, spot the trapezoid, and use convexity to nail the infimum.

Example: The Trapezoid Rule

Let $f(a,b) = b^3 - a^3 + a^2b - ab^2$. *There exists a real number* C *such that, regardless of the choice of* $0 = x_0 < x_1 < \cdots < x_n = 4$,

$$C \leq \sum_{i=1}^{n} f(x_{i-1}, x_i).$$

Compute the maximum value of C. [4]

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Try it: The Trapezoid Rule

Let $f(a,b) = b^3 - a^3 + a^2b - ab^2$. For $0 = x_0 < x_1 < \cdots < x_n = 4$, consider $\sum_{i=1}^{n} f(x_{i-1}, x_i)$.

Try factoring $f(a,b)$ — what does $(b-a)$ times something remind you of? [5]

[5] BMT 2023/5

Solution: The Trapezoid Rule

Factor, spot the trapezoid, and let convexity do the bounding.

Why factor: a factor $(b-a) = x_i - x_{i-1}$ is a *width*; whatever multiplies it is a *height*.

Factor: $f(a,b) = (b-a)(a^2+b^2)$, via the grouping

$$(b^3 - a^3) - ab(b-a) = (b-a)(a^2 + ab + b^2 - ab) = (b-a)(a^2+b^2).$$

Hence

$$\sum_{i=1}^{n} f(x_{i-1}, x_i) = 2\sum_{i=1}^{n} \frac{x_{i-1}^2 + x_i^2}{2}\,(x_i - x_{i-1}),$$

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which is twice the trapezoidal Riemann sum for x^2 on $[0,4]$: the height $\frac{a^2+b^2}{2}$ is the average of x^2 at the two ends. [6]

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Solution: The Trapezoid Rule (cont.)

Why convexity: the partition is arbitrary, so we cannot compute the sum — but we can compare it with the integral.

Since x^2 is convex, each chord lies above the graph, so each trapezoidal sum overestimates the integral:

$$\sum_{i=1}^{n} \frac{x_{i-1}^2 + x_i^2}{2}(x_i - x_{i-1}) \geq \int_0^4 x^2\,dx = \frac{64}{3},$$

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and the sums approach $\frac{64}{3}$ as the partition gets finer, so no larger bound works.

The greatest lower bound of the sums is therefore $2 \cdot \frac{64}{3}$, and the maximum admissible C is $\boxed{\frac{128}{3}}$. [7]

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Practice Exercises

Two moves, two exercises — and the answers.

Practice Exercise 1

Compute

$$\lim_{n\to\infty} \frac{1}{n^2} \int_0^n \lfloor x \rfloor \, dx.$$

Hint: As before, take n an integer. On $[k, k+1)$ the integrand is the constant k; sum $k = 0, \ldots, n-1$.

Practice Exercise 2

Let f be convex on $[0,1]$. For a partition $0 = x_0 < \cdots < x_n = 1$, is the trapezoidal sum

$$\sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}(x_i - x_{i-1})$$

always $\geq \int_0^1 f$, always $\leq$, or neither?

Hint: On each subinterval, where does the chord lie relative to a convex graph?

Practice: answers

- Exercise 1: $\lim_{n\to\infty} \frac{1}{n^2} \int_0^n \lfloor x \rfloor \, dx = \boxed{\frac{1}{2}}$
- Exercise 2: the trapezoidal sum is always $\geq \int_0^1 f$