Riemann Sums in Disguise

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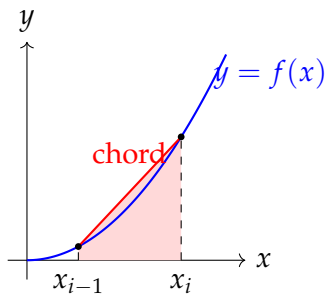
When a floor integral or a partition sum shows up, rewrite it as a Riemann sum and read off the limit — or compare it with the integral via convexity.

Floors become sums. On each unit interval $[k, k+1)$ the floor is constant, so $\int_0^n (\text{floor stuff})\, dx$ turns into a sum over k; lower-order terms like $O(1/n)$ vanish in the limit, so read off the Riemann-sum shape and drop the rest.

Trapezoids vs. integrals. A sum of the form

$$\sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} (x_i - x_{i-1})$$

is a *trapezoidal Riemann sum*: each term is the area of the trapezoid under the chord joining two points of the graph. For convex f the chord always lies above the graph, so every trapezoid overestimates its piece of the integral — each trapezoidal sum is $\geq \int f$, and the infimum over all partitions is the integral itself.



Problem 1 (BMT 2024/5)**.** For a real number n, let $\lfloor n \rfloor$ be the greatest integer less than or equal to n. Compute

$$\lim_{n\to\infty} \frac{1}{n^3} \int_0^n x\lfloor x \rfloor \, dx.$$

Solution. It suffices to take n a positive integer: for real n the leftover piece $\int_{\lfloor n \rfloor}^n x\lfloor x \rfloor \, dx$ is at most $n \cdot n \cdot 1 = n^2$, which is negligible next to n^3, and $\lfloor n \rfloor^3 / n^3 \to 1$.

On each unit interval $[k, k+1)$, $\lfloor x \rfloor = k$ is constant, so the integral is a sum:

$$\int_0^n x\lfloor x \rfloor \, dx = \sum_{k=0}^{n-1} \int_k^{k+1} kx \, dx = \sum_{k=0}^{n-1} k \cdot \frac{(k+1)^2 - k^2}{2} = \frac{1}{2} \sum_{k=0}^{n-1} k(2k+1).$$

Dividing by n^3 and splitting $k(2k+1) = 2k^2 + k$ to expose the Riemann-sum shape,

$$\frac{1}{n^3} \cdot \frac{1}{2} \sum_{k=0}^{n-1} k(2k+1) = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^2 + \frac{1}{2n^3} \sum_{k=0}^{n-1} k \longrightarrow \int_0^1 x^2 \, dx = \frac{1}{3},$$

because $\sum_{k=0}^{n-1} k < n^2$ makes the second term less than $\frac{1}{2n}$. The floor turned the integral into a sum, and the sum was a Riemann sum all along. □

Problem 2 (BMT 2023/5)**.** Let $f(a,b) = b^3 - a^3 + a^2 b - ab^2$. There exists a real number C such that, regardless of the choice of $0 = x_0 < x_1 < \cdots < x_n = 4$,

$$C \leq \sum_{i=1}^{n} f(x_{i-1}, x_i).$$

Compute the maximum value of C.

Solution. Factor: $f(a,b) = (b-a)(a^2+b^2)$. Hence

$$\sum_{i=1}^{n} f(x_{i-1}, x_i) = 2 \sum_{i=1}^{n} \frac{x_{i-1}^2 + x_i^2}{2} (x_i - x_{i-1}),$$

which is twice the trapezoidal Riemann sum for x^2 on $[0,4]$. Since x^2 is convex, each such sum overestimates the integral:

$$\sum_{i=1}^{n} \frac{x_{i-1}^2 + x_i^2}{2} (x_i - x_{i-1}) \geq \int_0^4 x^2 \, dx = \frac{64}{3},$$

with equality in the limit of fine partitions. The greatest lower bound of the sums is therefore $2 \cdot \frac{64}{3} = \frac{128}{3}$, and the maximum admissible C is $\frac{128}{3}$. □

Exercise 3. Compute $\lim_{n\to\infty} \frac{1}{n^2} \int_0^n \lfloor x \rfloor \, dx$. Hint: 1

Exercise 4. Let f be convex on $[0,1]$. For a partition $0 = x_0 < \cdots < x_n = 1$, is the trapezoidal sum $\sum \frac{f(x_{i-1}) + f(x_i)}{2} (x_i - x_{i-1})$ always $\geq \int_0^1 f$, always $\leq$, or neither? Hint: 2

Answers: $\frac{1}{3}$; $\frac{128}{3}$; $\frac{1}{2}$; always $\geq$.

Hints

1. As in the first problem, it suffices to take n an integer. On $[k, k+1)$ the integrand is the constant k. Sum $k = 0, \ldots, n-1$.
2. On each subinterval, where does the chord lie relative to a convex graph?