Self-Reference and Fixed Points

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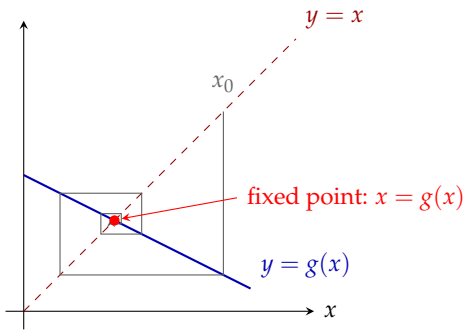

Self-reference in action: the iterates $x_{n+1} = g(x_n)$ feed into themselves and land on $x = g(x)$.

Self-Consistency: Write the Equation

When an object is defined in terms of itself, write the self-consistency equation and solve for it directly — never unwind the definition.

Self-Reference: The Move

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How it works

If an unknown quantity (often a derivative value like $f'(2)$) appears inside the definition of the function itself, differentiate the defining equation to get a second relation involving that same quantity, evaluate at the right point, and solve for it first. Only then compute the requested value.

Example: Self-Reference

Problem

Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies

$$f(x) = f'(2)x^2 + x \qquad \text{for all } x.$$

Then $f(2) = \frac{m}{n}$ with $\gcd(m, n) = 1$. Compute $m + n$.

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[1] BMT 2020/3

Try it: Self-Reference

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$$f(x) = f'(2)x^2 + x \qquad \text{for all } x.$$

Then $f(2) = \frac{m}{n}$ with $\gcd(m, n) = 1$. Compute $m + n$.

Hint: The unknown number $f'(2)$ sits inside the definition of f — what second equation can you get by differentiating?

Note: plugging $x = 2$ into the definition gives $f(2) = 4f'(2) + 2$ — one equation, two unknowns. [2]

[2] BMT 2020/3

Solution: Self-Reference

The unknown constant $f'(2)$ appears inside the definition of f itself.

Why differentiate: plugging in $x = 2$ gives one equation in two unknowns; differentiating gives a second equation in the *same* unknown $f'(2)$.

$$f'(x) = 2f'(2)x + 1.$$

[3] BMT 2020/3

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$$f'(x) = 2f'(2)x + 1.$$

Why $x = 2$: that is the point where the left side becomes $f'(2)$ again — the self-consistency equation.

$$f'(2) = 4f'(2) + 1 \implies -3f'(2) = 1 \implies f'(2) = -\frac{1}{3}.$$

[3] BMT 2020/3

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Only now compute what was asked: $f(2) = 4f'(2) + 2 = -\frac{4}{3} + 2 = \frac{2}{3}$, so $m + n = 2 + 3 = \boxed{5}$. [3]

[3] BMT 2020/3

Fixed Points of Recurrences

If $x_{n+1} = g(x_n)$ converges, take the limit through the continuous function g: the limit satisfies $x = g(x)$ — and a strictly decreasing g meets $y = x$ at most once.

Iterates and Fixed Points: The Move

If $x_{n+1} = g(x_n)$ converges to x, take limits on both sides: the left tends to x and (by continuity) the right tends to $g(x)$, so the limit must satisfy the fixed-point equation $x = g(x)$ — never unwind the iteration.

Uniqueness via monotonicity

If g is strictly *decreasing*, then $y = g(x)$ crosses $y = x$ at most once, so the fixed point is unique.

Example: Iterates and Fixed Points

Problem

Let $x_1 = -4$, and for $n \geq 1$ define $x_{n+1} = -4^{x_n}$. Similarly, let $f_1(x) = \sin(\arccos x)$, and for $n \geq 1$ define $f_{n+1}(x) = f_1(f_n(x))$. Compute

$$\lim_{n \to \infty} f_n(2^{x_n}).$$

You may assume that this limit exists.

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[4] BMT 2021/6

Try it: Iterates and Fixed Points

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You may assume that this limit exists.

Hint: If x_n converges, what equation must the limit satisfy? Solve it by guessing. [5]

[5] BMT 2021/6

Solution: Iterates and Fixed Points (1/2)

Why take limits: we never compute x_{100}; if $x_n \to x$, the recurrence forces $x = g(x)$ with $g(x) = -4^x$.

$$x = -4^x.$$

(We assume x_n converges, as intended. Numerically $x_{20} \approx -0.4992$, and $|g'(-\frac{1}{2})| = \ln 2 < 1$, so steps near $-\frac{1}{2}$ shrink the distance to it.)

[6] BMT 2021/6

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Since every iterate is negative, try a simple negative value: $x = -\frac{1}{2}$ works, because $-4^{-1/2} = -\frac{1}{2}$. It is the only solution because $x \mapsto -4^x$ is strictly decreasing.

[6] BMT 2021/6

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$$2^{x_n} \longrightarrow 2^{-1/2} = \frac{1}{\sqrt{2}}.$$

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[6] BMT 2021/6

Solution: Iterates and Fixed Points (2/2)

So far: $2^{x_n} \to \frac{1}{\sqrt{2}}$. Every x_n is negative, so $2^{x_n} \in (0,1)$ for all n — that is where $f_1(x) = \sqrt{1-x^2}$ applies.

Now $\frac{1}{\sqrt{2}}$ is itself a fixed point of f_1:

$$f_1\left(\tfrac{1}{\sqrt{2}}\right) = \sin\left(\arccos \tfrac{1}{\sqrt{2}}\right) = \sin \frac{\pi}{4} = \tfrac{1}{\sqrt{2}},$$

so $f_n\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ for every n.

[7] BMT 2021/6

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$$\lim_{n\to\infty} f_n(2^{x_n}) = \boxed{\frac{1}{\sqrt{2}}}.$$

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[7] BMT 2021/6

Practice Exercises

1. Suppose g is differentiable with $g(x) = ax^2 + x$, where $a = g'(1)$. Find $g(2)$.
2. Suppose h is differentiable with $h(x) = dx^3 + x$, where $d = h'(2)$. Find $h(1)$.

Practice Exercise Answers

1. $g'(x) = 2ax + 1$, so $2a + 1 = a$, $a = -1$; $g(2) = \boxed{-2}$

2. $h'(x) = 3dx^2 + 1$, so $12d + 1 = d$, $d = -\frac{1}{11}$; $h(1) = \boxed{\frac{10}{11}}$