Self-Reference and Fixed Points

David Liu

When an object is defined in terms of itself, write the self-consistency equation and solve for it directly — never unwind the definition.

Problem 1 (BMT 2020/3)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $f(x) = f'(2)x^2 + x$ for all x. Then $f(2) = \frac{m}{n}$ with $\gcd(m, n) = 1$. Compute $m + n$.

Solution. The unknown constant $f'(2)$ appears inside the definition of f itself. Differentiate both sides to get a second relation involving it:

$$f'(x) = 2f'(2)x + 1.$$

Evaluating at $x = 2$ pins down the constant:

$$f'(2) = 4f'(2) + 1 \quad \Longrightarrow \quad f'(2) = -\frac{1}{3}.$$

Then $f(2) = 4f'(2) + 2 = -\frac{4}{3} + 2 = \frac{2}{3}$, so $m + n = 2 + 3 = 5$. □

Recurrences that settle down. If $x_{n+1} = g(x_n)$ converges to x, take limits on both sides: the left tends to x and (by continuity) the right tends to $g(x)$, so the limit must satisfy the fixed-point equation $x = g(x)$ — never unwind the iteration. If g is strictly decreasing, $y = g(x)$ crosses $y = x$ at most once, so the fixed point is unique.

Problem 2 (BMT 2021/6)**.** Let $x_1 = -4$, and for $n \geq 1$ define $x_{n+1} = -4^{x_n}$. Similarly, let $f_1(x) = \sin(\arccos x)$, and for $n \geq 1$ define $f_{n+1}(x) = f_1(f_n(x))$. Compute $\lim_{n\to\infty} f_n(2^{x_n})$. You may assume that this limit exists.

Solution. The recurrence settles at a fixed point. (We take for granted that x_n converges, as the problem intends. Numerically it does: $x_{10} \approx -0.470$, $x_{20} \approx -0.4992$, $x_{30} \approx -0.49998$. Near $-\frac{1}{2}$ the map $g(x) = -4^x$ has slope of size $|g'(-\frac{1}{2})| = \ln 4 \cdot 4^{-1/2} = \ln 2 < 1$, so each step shrinks the distance to the fixed point.) Taking limits in $x_{n+1} = -4^{x_n}$ gives

$$x = -4^x.$$

Since every iterate is negative, try a simple negative value: $x = -\frac{1}{2}$ works, because $-4^{-1/2} = -\frac{1}{2}$. It is the only solution because $x \mapsto -4^x$ is strictly decreasing. Hence $2^{x_n} \to 2^{-1/2} = \frac{1}{\sqrt{2}}$.

Next, $\frac{1}{\sqrt{2}}$ is itself a fixed point of f_1:

$$f_1\left(\tfrac{1}{\sqrt{2}}\right) = \sin\left(\arccos \tfrac{1}{\sqrt{2}}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

so $f_n\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$ for every n. Every x_n is negative, so $2^{x_n} \in (0,1)$, and there $f_1(x) = \sqrt{1-x^2}$ (because $\arccos x \in [0, \frac{\pi}{2}]$, where sine is nonnegative). Since $\sqrt{1-x^2}$ maps $[0,1]$ to itself and undoes itself, f_n alternates between $\sqrt{1-x^2}$ (odd n) and x (even n); both tend to $\frac{1}{\sqrt{2}}$ as $2^{x_n} \to \frac{1}{\sqrt{2}}$. Therefore

$$\lim_{n\to\infty} f_n(2^{x_n}) = \frac{1}{\sqrt{2}}.$$

□

Exercise 3. Suppose g is differentiable with $g(x) = ax^2 + x$, where $a = g'(1)$. Find $g(2)$. Hint: 1

Exercise 4. Suppose h is differentiable with $h(x) = dx^3 + x$, where $d = h'(2)$. Find $h(1)$. Hint: 2

Answers: 5; $1/\sqrt{2}$; -2; $10/11$.

Hints

1. Differentiate, evaluate at $x = 1$, and solve for a first.
2. Differentiate, evaluate at $x = 2$, and solve for d first.