Tangency and Multiple Roots

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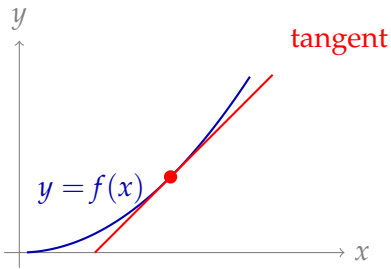

One point of contact: the curves share a value *and* a slope there.

Tangency as a Double Root

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Suppose $y = f(x)$ and $y = g(x)$ meet at x_0 and $f - g$ keeps the *same sign* on both sides of x_0 (the curves touch rather than cross). Then x_0 is a minimum or maximum of $f - g$, so its derivative vanishes:

$$f(x_0) = g(x_0) \qquad \text{and} \qquad f'(x_0) = g'(x_0).$$

This is two equations in the unknowns x_0 and the parameter.

Tangency: Match Values and Derivatives

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Careful: one meeting point alone is not enough ($y = x$ and $y = 0$ cross once). Check the sign of $f - g$ at both ends.

Example: Match Values and Derivatives

Find the positive real value of k for which $e^{kx} = 3\sqrt{x}$ has exactly one solution. [1]

[1] BMT 2024/4

Try it: Match Values and Derivatives

Find the positive real value of k for which $e^{kx} = 3\sqrt{x}$ has exactly one solution.

One intersection of the two curves — what must their slopes satisfy there? [2]

[2] BMT 2024/4

Solution: Match Values and Derivatives

Why tangency: $e^{kx} - 3\sqrt{x} \to 1 > 0$ as $x \to 0^+$ and $\to \infty$ as $x \to \infty$, so a single zero must be a touch, not a crossing — the curves are tangent there: values and derivatives agree.

$$e^{kx} = 3\sqrt{x}, \qquad ke^{kx} = \frac{3}{2\sqrt{x}}.$$

[3] BMT 2024/4

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Why substitute: the exponential is the hard part, and the first equation replaces it by $3\sqrt{x}$:

$$k \cdot 3\sqrt{x} = \frac{3}{2\sqrt{x}} \quad \Longrightarrow \quad 2kx = 1,$$

so the contact point is $x = \frac{1}{2k}$. [3]

[3] BMT 2024/4

Solution: Match Values and Derivatives (cont.)

So far: tangency forces the contact point $x = \frac{1}{2k}$.

[4] BMT 2024/4

Solution: Match Values and Derivatives (cont.)

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Plugging the contact point back into the value equation pins down k:

$$e^{1/2} = 3\sqrt{\frac{1}{2k}} \quad \Longrightarrow \quad \sqrt{2ek} = 3,$$

hence $\boxed{k = \frac{9}{2e}}$. [4]

[4] BMT 2024/4

Common Roots of f and f'

A common root of f and f' is at least a double root of f: use it to factor.

Common Roots of f and f': The Move

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Proof: write $f(x) = (x - a)g(x)$.

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$$f(x) = (x-a)^2 g_1(x).$$

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On contest problems this lets you factor f immediately.

Example: Common Roots of f and f'

Let $f(x)$ be a degree-4 polynomial such that $f(x)$ and $f'(x)$ both have 20 and 22 as roots. Given $f(21) = 21$, compute $f(23)$. [5]

[5] BMT 2022/4

Try it: Common Roots of f and f'

Let $f(x)$ be a degree-4 polynomial such that $f(x)$ and $f'(x)$ both have 20 and 22 as roots. Given $f(21) = 21$, compute $f(23)$.

f and f' share the roots 20 and 22 — what does that force f to look like? [6]

[6] BMT 2022/4

Solution: Common Roots of f and f'

Both 20 and 22 are common roots of f and f', so each is at least a double root:

$$f(x) = (x-20)^2(x-22)^2 \cdot r(x).$$

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Therefore $\boxed{f(23) = 21 \cdot 3^2 \cdot 1^2 = 189}$.

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Why on $y = x$: a meeting point (z, w) means $f(z) = w$ and $f(w) = z$. If $z < w$, increasing gives $w = f(z) < f(w) = z$ — impossible. So $z = w$.

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Why slope 1: the slopes there are $f'(z)$ and $1/f'(z)$; they agree only if $f'(z) = \pm 1$, and $f' > 0$ leaves 1.

Example: Tangency of Inverse Functions

For some $a > 1$, the curves $y = a^x$ and $y = \log_a x$ are tangent to each other at exactly one point. Compute $|\ln(\ln a)|$. [7]

[7] BMT 2020/6

Try it: Tangency of Inverse Functions

For some $a > 1$, the curves $y = a^x$ and $y = \log_a x$ are tangent to each other at exactly one point. Compute $|\ln(\ln a)|$.

Where can inverse curves meet tangentially? What does that force the common tangent to be? [8]

[8] BMT 2020/6

Solution: Tangency of Inverse Functions (1/2)

Since a^x and $\log_a x$ are inverses and a^x is increasing ($a > 1$), the tangency point lies on $y = x$; call it (z, z). Then

$$a^z = \log_a z.$$

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The common tangent at (z, z) is $y = x$ itself (slope -1 is ruled out since $a > 1$ makes both derivatives positive), so both derivatives equal 1 there — two equations for two unknowns z, a:

$$a^z \ln a = 1, \qquad \frac{1}{z \ln a} = 1.$$

Solution: Tangency of Inverse Functions (2/2)

From $a^z = \log_a z$ and $a^z \ln a = 1$ we get $\ln a \cdot \log_a z = 1$, i.e. $\ln z = 1$, so

$$z = e.$$

Solution: Tangency of Inverse Functions (2/2)

From $a^z = \log_a z$ and $a^z \ln a = 1$ we get $\ln a \cdot \log_a z = 1$, i.e. $\ln z = 1$, so

$$z = e.$$

Then $e \ln a = 1$ gives $\ln a = \frac{1}{e}$, and

$$\boxed{|\ln(\ln a)| = \left|\ln \frac{1}{e}\right| = 1}.$$

Practice Exercises

Try them with the three moves above.

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1. Find the positive real k for which $e^{kx} = x$ has exactly one (positive) solution.
   Hint: match values and derivatives at the contact point, then eliminate x.

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1. Find the positive real k for which $e^{kx} = x$ has exactly one (positive) solution. *Hint: match values and derivatives at the contact point, then eliminate x.*

2. Let g be a degree-2 polynomial with 1 a root of both g and g', and $g(0) = 1$. Compute $g(2)$. *Hint: the common root is a double root, so $g(x) = c(x-1)^2$. Find c.*

Practice Exercise Answers

1. $\frac{1}{e}$ (match values and derivatives, eliminate x)

2. 1 ($g(x) = c(x-1)^2$, $g(0) = 1$ gives $c = 1$)