Tangency and Multiple Roots

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"Exactly one solution" means the curves are tangent: match values *and* derivatives. A common root of f and f' is a multiple root — use it to factor.

Mini-move (inverse symmetry): if f is increasing and meets f^{-1} tangentially, the contact point lies on $y = x$ and the common tangent is $y = x$ itself. Why: if $f(z) = w$ and $f(w) = z$ with $z < w$, then increasing gives $w = f(z) < f(w) = z$, a contradiction; so every meeting point has $z = w$. There the slopes are $f'(z)$ and $1/f'(z)$, which agree only when $f'(z) = \pm 1$, and $f' > 0$ leaves 1.

Problem 1 (BMT 2024/4)**.** Find the positive real value of k for which $e^{kx} = 3\sqrt{x}$ has exactly one solution.

Solution. The difference $e^{kx} - 3\sqrt{x}$ tends to $1 > 0$ as $x \to 0^+$ and to ∞ as $x \to \infty$, so it is positive at both ends. A single zero therefore must be a touch, not a crossing: the difference keeps the same sign on both sides, so the zero is a minimum of the difference and its derivative vanishes there. The curves $y = e^{kx}$ and $y = 3\sqrt{x}$ are tangent: both the values and the derivatives agree.

$$e^{kx} = 3\sqrt{x}, \qquad ke^{kx} = \frac{3}{2\sqrt{x}}.$$

Substituting the first equation into the second eliminates the exponential:

$$k \cdot 3\sqrt{x} = \frac{3}{2\sqrt{x}} \quad \Longrightarrow \quad 2kx = 1,$$

so the contact point is $x = \frac{1}{2k}$. Plugging back into $e^{kx} = 3\sqrt{x}$:

$$e^{1/2} = 3\sqrt{\frac{1}{2k}} \quad \Longrightarrow \quad \sqrt{2ek} = 3,$$

hence $k = \frac{9}{2e}$. □

Problem 2 (BMT 2022/4)**.** Let $f(x)$ be a degree-4 polynomial such that $f(x)$ and $f'(x)$ both have 20 and 22 as roots. Given $f(21) = 21$, compute $f(23)$.

Solution. If $f(a) = f'(a) = 0$, write $f(x) = (x - a)g(x)$; then $f'(x) = (x - a)g'(x) + g(x)$, so $f'(a) = 0$ forces $g(a) = 0$. Thus a is at least a double root.

Applying this at $a = 20$ and $a = 22$ gives $f(x) = (x - 20)^2(x - 22)^2 \cdot r(x)$. Since $\deg f = 4$, $r(x) = c$ is constant. From $f(21) = 21$:

$$c \cdot 1^2 \cdot (-1)^2 = 21 \quad \Longrightarrow \quad c = 21.$$

Therefore $f(23) = 21 \cdot 3^2 \cdot 1^2 = 189$. □

Problem 3 (BMT 2020/6)**.** For some $a > 1$, the curves $y = a^x$ and $y = \log_a x$ are tangent to each other at exactly one point. Compute $|\ln(\ln a)|$.

Solution. Since a^x and $\log_a x$ are inverse functions and a^x is increasing (as $a > 1$), the mini-move puts their point of tangency on the line $y = x$. Call it (z, z); then $a^z = \log_a z$.

The common tangent at a point of the symmetry line is the line $y = x$ itself (slope -1 is ruled out since $a > 1$ makes both derivatives positive), so both derivatives equal 1 there:

$$a^z \ln a = 1, \qquad \frac{1}{z \ln a} = 1.$$

From $a^z = \log_a z$ and $a^z \ln a = 1$ we get $\ln a \cdot \log_a z = 1$, i.e. $\ln z = 1$, so $z = e$. Then $e \ln a = 1$ gives $\ln a = \frac{1}{e}$, and

$$|\ln(\ln a)| = \left|\ln \frac{1}{e}\right| = 1.$$

□

Exercise 4. Find the positive real k for which $e^{kx} = x$ has exactly one (positive) solution. Hint: 1

Exercise 5. Let g be a degree-2 polynomial with 1 a root of both g and g', and $g(0) = 1$. Compute $g(2)$. Hint: 2

Answers: $\frac{9}{2e}$; 189; 1; $\frac{1}{e}$; 1.

Hints

1. Match values and derivatives at the contact point, then eliminate x.
2. The common root is a double root, so $g(x) = c(x-1)^2$. Find c.