Telescoping and Collapsing

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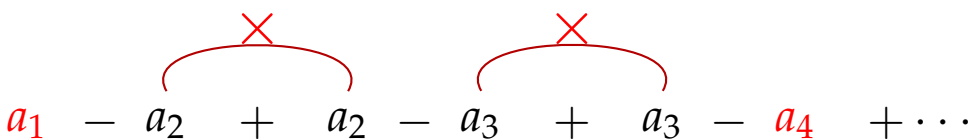

Everything in the middle cancels: the sum collapses to $a_1 - a_4$.

Telescoping Sums

Rewrite terms as differences so the sum collapses; rationalize conjugates to expose the surviving limit.

Telescoping Sums: The Move

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The move

Write each term as a difference $f(n+1)-f(n)$ (or $g(n)-g(n+1)$). Then the partial sum telescopes:

$$\sum_{n=0}^{k}\big(f(n+1)-f(n)\big)=f(k+1)-f(0).$$

If a square-root difference survives, rationalize the conjugate to read off the limit.

Example: Telescoping Sums

Compute

$$\sum_{n=0}^{\infty} \left(\sqrt{n^2 + 3n + 2} - \sqrt{n^2 + n} - 1 \right).$$

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[1] BMT 2022/5

Try it: Telescoping Sums

Compute

$$\sum_{n=0}^{\infty} \left(\sqrt{n^2 + 3n + 2} - \sqrt{n^2 + n} - 1 \right).$$

Hint: factor inside the roots so the two sums have the same shape.

Solution: Telescoping Sums

Why factor first: the two radicands must have the same shape, shifted by one, for anything to cancel: $\sqrt{(n+1)(n+2)} - \sqrt{n(n+1)}$.

[2] BMT 2022/5

Solution: Telescoping Sums

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Why partial sums: telescoping needs a finite sum; take $k \to \infty$ at the end.

$$\begin{aligned} S_k &= \sum_{n=0}^{k} \sqrt{(n+1)(n+2)} - \sum_{n=0}^{k} \sqrt{n(n+1)} - (k+1) \\ &= \sum_{n=1}^{k+1} \sqrt{n(n+1)} - \sum_{n=0}^{k} \sqrt{n(n+1)} - (k+1) \end{aligned}$$

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the sums telescope: only the top term of the first and the bottom term $\sqrt{0 \cdot 1} = 0$ of the second survive. [2]

[2] BMT 2022/5

The Limit: Telescoping Sums

Rationalize the conjugate:

$$S_k = \frac{(k+1)(k+2) - (k+1)^2}{\sqrt{(k+1)(k+2)} + (k+1)} = \frac{k+1}{\sqrt{(k+1)(k+2)} + (k+1)}$$
$$= \frac{1}{\sqrt{(k+2)/(k+1)} + 1} \to \frac{1}{2},$$

after dividing top and bottom by $(k+1)$.

Why rationalize: $\sqrt{(k+1)(k+2)} - (k+1)$ is "$\infty - \infty$"; the conjugate turns it into a quotient we can read off.

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Hence

$$\lim_{k\to\infty} S_k = \frac{1}{2}, \qquad \text{so the sum is } \boxed{\tfrac{1}{2}}.$$

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[3] BMT 2022/5

The Collapsing Combination

For recurrences, find the combination that shrinks geometrically.

The Collapsing Combination: The Move

For recurrences, find the combination that shrinks geometrically.

The move

When a recurrence couples two sequences, don't track them separately — try simple combinations (product, ratio, difference) until one of them has a clean recurrence of its own. It usually shrinks geometrically, and the limit follows. Here $b_{n+1} = a_n b_n / (a_n + b_n)$ (half the harmonic mean) has the sum $a_n + b_n$ in its denominator, and the arithmetic mean a_{n+1} has the same sum on top, so multiplying them cancels it: try the product first.

Example: The Collapsing Combination

Let $a_{n+1} = \dfrac{a_n + b_n}{2}$ *and* $b_{n+1} = \dfrac{1}{\frac{1}{a_n} + \frac{1}{b_n}}$, *with* $a_0 = 13$ *and* $b_0 = 29$. *What is* $\lim_{n\to\infty} a_n b_n$?

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[4] BMT 2018/2

Solution: The Collapsing Combination

Track the *product* instead of the two sequences separately. *Why:* the sum $a_n + b_n$ is on top in a_{n+1} and on the bottom in b_{n+1}.

$$a_{n+1}b_{n+1} = \frac{a_n + b_n}{2} \cdot \frac{a_n b_n}{a_n + b_n} = \frac{a_n b_n}{2}.$$

[5] BMT 2018/2

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[5] BMT 2018/2

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$$a_n b_n = \frac{a_0 b_0}{2^n} = \frac{13 \cdot 29}{2^n} \longrightarrow \boxed{0}.$$

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[5] BMT 2018/2

Practice Exercises

Practice: apply the moves yourself.

Practice Exercises (1/2)

Compute

$$\sum_{n=1}^{99} \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Hint: rationalize the denominator to get a difference of square roots.

Practice Exercises (2/2)

Let $p_0 = 1$ *and* $p_{n+1} = \dfrac{p_n}{1 + p_n}$. *Compute* $\lim_{n \to \infty} p_n$.

Hint: consider $\frac{1}{p_{n+1}} - \frac{1}{p_n}$*; it telescopes.*

Practice Exercise Answers

1. 9 (rationalize $\to$ difference of roots, telescope)

2. 0 ($1/p_{n+1} - 1/p_n = 1$ telescopes to $1/p_n = n + 1$)