Telescoping and Collapsing

David Liu

Rewrite terms as differences so the sum collapses; rationalize conjugates to expose the surviving limit. For recurrences, find the combination that shrinks geometrically.

Problem 1 (BMT 2022/5)**.** Compute

$$\sum_{n=0}^{\infty} \left(\sqrt{n^2+3n+2} - \sqrt{n^2+n} - 1 \right).$$

Solution. Factor inside the roots: $\sqrt{(n+1)(n+2)} - \sqrt{n(n+1)}$. Let S_k be the k-th partial sum:

$$\begin{aligned} S_k &= \sum_{n=0}^{k} \sqrt{(n+1)(n+2)} - \sum_{n=0}^{k} \sqrt{n(n+1)} - (k+1) \\ &= \sum_{n=1}^{k+1} \sqrt{n(n+1)} - \sum_{n=0}^{k} \sqrt{n(n+1)} - (k+1) \\ &= \sqrt{(k+1)(k+2)} - (k+1), \end{aligned}$$

since the two sums telescope, leaving only the top term of the first sum and the bottom term $\sqrt{0 \cdot 1} = 0$ of the second. Rationalize the conjugate for the limit:

$$S_k = \frac{(k+1)(k+2) - (k+1)^2}{\sqrt{(k+1)(k+2)} + (k+1)} = \frac{k+1}{\sqrt{(k+1)(k+2)} + (k+1)} \longrightarrow \frac{1}{2}.$$

□

The collapsing combination. When a recurrence couples two sequences, don't track them separately — try simple combinations (product, ratio, difference) until one of them has a clean recurrence of its own. It usually shrinks geometrically, and the limit follows.

Problem 2 (BMT 2018/2)**.** Let $a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = \frac{1}{\frac{1}{a_n} + \frac{1}{b_n}}$, with $a_0 = 13$ and $b_0 = 29$. What is $\lim_{n\to\infty} a_n b_n$?

Solution. Instead of tracking a_n and b_n separately, look at their product. Why the product: $b_{n+1} = \frac{a_n b_n}{a_n + b_n}$ has $a_n + b_n$ in its denominator, exactly the sum that appears in a_{n+1}, so multiplying cancels it:

$$a_{n+1} b_{n+1} = \frac{a_n + b_n}{2} \cdot \frac{a_n b_n}{a_n + b_n} = \frac{a_n b_n}{2}.$$

The product halves every step — it collapses geometrically. Hence $a_n b_n = \frac{a_0 b_0}{2^n} \to 0$. □

Exercise 3. Compute $\displaystyle\sum_{n=1}^{99} \frac{1}{\sqrt{n+1}+\sqrt{n}}$. Hint: 1

Exercise 4. Let $p_0 = 1$ and $p_{n+1} = \frac{p_n}{1+p_n}$. Compute $\lim_{n\to\infty} p_n$. Hint: 2

Answers: $\frac{1}{2}$; 0; 9; 0.

Hints

1. Rationalize the denominator to get a difference of square roots.
2. Consider $\frac{1}{p_{n+1}} - \frac{1}{p_n}$; it telescopes.