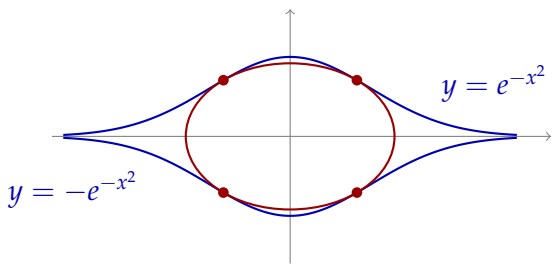
Hard Problems

David Liu



The largest circle tangent to both curves: BMT 2018/8

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First-Step Conditioning

Set up the expected-value equation from first-step analysis, then differentiate it into an ODE.

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First-step conditioning: the technique

Set up the expected-value equation from first-step analysis, then differentiate it into an ODE.

1. Let $f(x)$ = expected remaining steps from x.
2. Condition on the first step: it costs 1, then continue from the new total.
3. Average over U: $f(x) = 1 + \frac{1}{M}\int_x^{\text{threshold}} f(t)\,dt$; differentiate (FTC) $\to$ ODE.

If the first item pushes the total past the threshold we stop (0 further items); otherwise we continue from the new total — hence the integral runs from x to the threshold.

First-Step Conditioning: Example

At the Berkeley Mart for Technology, every item has a real-number cost independently and uniformly distributed from 0 to 2022. Sumith buys different items at the store until the total amount he spends strictly exceeds 1. Compute the expected value of the number of items Sumith buys. [1]

[1] BMT 2022/8

First-Step Conditioning: Try it

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Let $f(x)$ be the expected remaining items starting from total x — what happens on the first draw? [2]

[2] BMT 2022/8

First-Step Conditioning: Solution (1/2)

Let $f(x)$ be the expected additional number of items given that x has already been spent; we want $f(0)$. One item moves the total to $x + U$ with U uniform on $[0, 2022]$, and we continue only while the total stays ≤ 1:

$$f(x) = 1 + \frac{1}{2022} \int_x^1 f(t)\, dt.$$

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$$f(x) = 1 + \frac{1}{2022} \int_x^1 f(t)\, dt.$$

Differentiate via the FTC:

$$f'(x) = -\frac{f(x)}{2022}.$$

First-Step Conditioning: Solution (2/2)

Then $f(x) = Ce^{-x/2022}$, and at $x = 1$ the integral vanishes, so $f(1) = 1$ — one more item always finishes the job. Hence $C = e^{1/2022}$.

Sensible: each item exceeds \$1 with probability 2021/2022, so the answer $e^{1/2022} \approx 1.0005$ is barely above 1.

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$$f(0) = e^{1/2022}.$$

Differentiate to Match

To match an integral to a proposed antiderivative, differentiate the answer and compare term by term.

Differentiate to match: the method

To match an integral to a proposed antiderivative, differentiate the answer and compare term by term.

Given $\int I(x)\,dx = F(x) + K$ with a proposed antiderivative F:

1. Differentiate F and simplify it into the same shape as I.
2. Match exponents (and coefficients) term by term to solve for the parameters.

The answer is given, so verify it in the easy direction — differentiation is mechanical, integration is not.

Differentiate to Match: Example

Suppose the following equality holds, where a, b, c are integers and K is the constant of integration:

$$\int \frac{\sin^a x - \cos^a x}{\sin^b x \cos^b x}\, dx = \frac{\csc^c x}{c} + \frac{\sec^c x}{c} + K.$$

If $a = 2021$, compute $a + b + c$. [3]

[3] BMT 2021/5

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If $a = 2021$, compute $a + b + c$.

The answer is given — which direction is easy to check: integration or differentiation? [4]

[4] BMT 2021/5

Differentiate to Match: Solution

Differentiating the proposed answer,

$$\frac{d}{dx}\left(\frac{\csc^c x}{c}+\frac{\sec^c x}{c}\right)=-\csc^c x\cot x+\sec^c x\tan x=-\frac{\cos x}{\sin^{c+1} x}+\frac{\sin x}{\cos^{c+1} x}.$$

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The integrand rewrites as

$$(\sin^a x-\cos^a x)/(\sin^b x\cos^b x) = \sin^a x/(\sin^b x\cos^b x)-\cos^a x/(\sin^b x\cos^b x),$$

which simplifies to $\frac{\sin^{a-b} x}{\cos^b x}-\frac{\cos^{a-b} x}{\sin^b x}$.

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Matching exponents term by term: $a-b=1$ and $b=c+1$. With $a=2021$: $b=2020$, $c=2019$, so

$$a+b+c=6060.$$

Complete the Square

To maximize an integral over functions, complete the square in the integrand — the maximizer kills the square pointwise.

Complete the Square: Example

Let $f : [1, \infty) \to \mathbb{R}$ be continuous such that

$$I(f) = \int_1^\infty \left(\sqrt{2023}\, x e^{-x} f(x) - \tfrac{1}{4} x^2 f(x)^2 \right) dx$$

converges. If $I(f)$ is maximized over all continuous f, compute $f(1) + I(f)$. [5]

[5] BMT 2023/8

Complete the square: the pointwise maximizer

To maximize an integral over functions, complete the square in the integrand — the maximizer kills the square pointwise.

At each fixed x, the integrand is a concave quadratic in the single number $f(x)$ — quadratics in one variable are maximized by completing the square, so do it pointwise in x.

$$-(axf(x) - be^{-x})^2 = 2ab\,xe^{-x}f(x) - a^2x^2f(x)^2 - b^2e^{-2x}.$$

Match coefficients with the integrand, kill the square.

Complete the Square: Try it

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At each fixed x, what kind of function of the single number $f(x)$ is the integrand? [6]

[6] BMT 2023/8

Complete the Square: Solution

Matching $2ab = \sqrt{2023}$ and $a^2 = \frac{1}{4}$ gives $a = \frac{1}{2}$ and $b = \sqrt{2023}$.

Hence

$$I(f) = \int_1^\infty \left[2023e^{-2x} - \left(\tfrac{1}{2} x f(x) - \sqrt{2023}\, e^{-x} \right)^2 \right] dx.$$

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The squared term is nonpositive, so $I(f)$ is maximized exactly when it vanishes pointwise:

$$f(x) = \frac{2\sqrt{2023}\, e^{-x}}{x}.$$

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Then

$$f(1) = \frac{2\sqrt{2023}}{e}, \qquad I(f) = \int_1^\infty 2023e^{-2x}\, dx = \frac{2023}{2e^2},$$

so $f(1) + I(f) = \frac{2\sqrt{2023}}{e} + \frac{2023}{2e^2}$.

Rescale to Stabilize

Before taking a limit at infinity, rescale coordinates so the geometry stabilizes.

Rescale to Stabilize: Example

For positive real numbers a, let the circle $C(a)$ be the graph of $x^2 + (y-a)^2 = 4a$ in the coordinate plane. Let $f(a)$ be the fraction of the area of $C(a)$ that lies above the parabola $y = x^2$. Find $\lim_{a\to\infty} f(a)$. [7]

[7] BMT 2025/7

Rescale to stabilize: the move

Before taking a limit at infinity, rescale coordinates so the geometry stabilizes.

The circle's center is $(0, a)$ and its radius is $2\sqrt{a}$ — translate to the center and divide by the radius; that is the natural normalization.

For the circle $x^2 + (y - a)^2 = 4a$, set

$$x = 2\sqrt{a}\,u, \qquad y = a + 2\sqrt{a}\,v.$$

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Then

- the circle becomes the fixed unit circle $u^2 + v^2 = 1$;
- the parabola $y = x^2$ becomes $a + 2\sqrt{a}\,v = 4au^2$.

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The circle's center and radius grow with a — what change of variables freezes them? [8]

[8] BMT 2025/7

Rescale to Stabilize: Solution

Why divide by $4a$*:* it shows which terms survive. Above the parabola means $u^2 \leq \frac{1}{4} + \frac{v}{2\sqrt{a}} \to \frac{1}{4}$, so in rescaled coordinates the region above the parabola becomes the strip $|u| \leq \frac{1}{2}$ of the unit disk.

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Why sectors: $u = \pm\frac{1}{2}$ meets $u^2 + v^2 = 1$ where $\cos\theta = \pm\frac{1}{2}$, i.e. at $\theta = \pm 60^\circ$ and $\pm 120^\circ$. The strip is two 60° sectors (60° to 120° and -120° to -60°, area $\frac{\pi}{6}$ each) plus two triangles (vertex at the center, base a chord of length $\sqrt{3}$, area $\frac{\sqrt{3}}{4}$ each):

$$\lim_{a\to\infty} f(a) = \frac{2\cdot\frac{\pi}{6} + 2\cdot\frac{\sqrt{3}}{4}}{\pi} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}.$$

Tangency Orthogonality

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If a circle centered at the origin is tangent to a curve at (x, y), the radial vector (x, y) must be perpendicular to the curve's tangent direction $(1, y')$:

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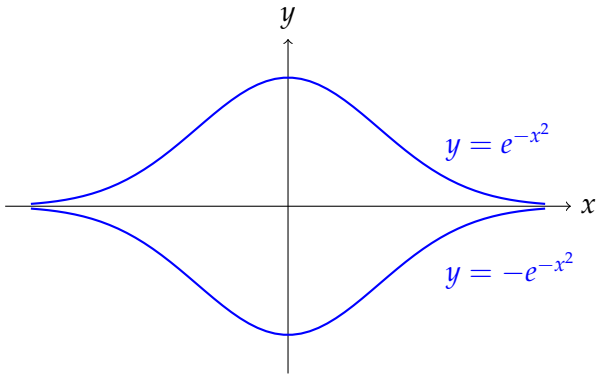
$$(x, y) \cdot (1, y') = x + y\,y' = 0.$$

Tangency Orthogonality: Example

What is the maximum radius of a circle tangent to the curves $y = e^{-x^2}$ and $y = -e^{-x^2}$ at two points each?



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[9] BMT 2018/8

Tangency Orthogonality: Try it

What is the maximum radius of a circle tangent to the curves $y = e^{-x^2}$ and $y = -e^{-x^2}$ at two points each?

Where must the circle be centered? What does tangency say about the radius vector? [10]

[10] BMT 2018/8

Tangency Orthogonality: Solution

The figure is symmetric in both x and y; we take the circle in the symmetric position the problem intends, between the curves and centered at the origin. Let (x, y) be a contact point on $y = e^{-x^2}$ with $x > 0$.

$x = 0$ gives the circle through $(0, \pm 1)$, tangent at just one point per curve — "two points each" forces $x \neq 0$.

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The tangent direction is $(1, y') = (1, -2xy)$, and orthogonality with (x, y) gives

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so $y = \frac{1}{\sqrt{2}}$ (as $x \neq 0$).

Then $e^{-x^2} = \frac{1}{\sqrt{2}}$ gives $x^2 = \frac{1}{2} \ln 2$, and

$$r^2 = x^2 + y^2 = \frac{\ln 2}{2} + \frac{1}{2} = \frac{1 + \ln 2}{2}, \qquad r = \sqrt{\frac{1 + \ln 2}{2}}.$$

Tangency Orthogonality: Why This Radius Is the Answer

Why look at distance: a circle about the origin is tangent to the curve exactly where the distance from the origin to the curve stops changing.

The squared distance to (x, e^{-x^2}) is $D = x^2 + e^{-2x^2}$. Put $t = x^2 \geq 0$:

$$D = t + e^{-2t}, \qquad \frac{dD}{dt} = 1 - 2e^{-2t}.$$

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$\frac{dD}{dt} < 0$ for $t < \frac{1}{2}\ln 2$ and > 0 after, so D is smallest at $x^2 = \frac{1}{2}\ln 2$ — our contact point.

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So $r = \sqrt{\frac{1+\ln 2}{2}}$ is the largest circle about the origin between the curves, touching each at $x = \pm\sqrt{\frac{1}{2}\ln 2}$. Other radii miss, touch only at $x = 0$, or cross — this is the unique two-point tangency, hence the maximum. [11]

[11] BMT 2018/8

Answers

All five final answers in one place.

The five answers

- BMT 2022/8: $e^{1/2022}$
- BMT 2021/5: 6060
- BMT 2023/8: $\frac{2\sqrt{2023}}{e} + \frac{2023}{2e^2}$
- BMT 2025/7: $\frac{1}{3} + \frac{\sqrt{3}}{2\pi}$
- BMT 2018/8: $\sqrt{\frac{1+\ln 2}{2}}$