Hard Problems

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Five advanced problems, each showcasing a technique worth knowing. Full solutions included.

Set up the expected-value equation from first-step analysis, then differentiate it into an ODE.

First-step conditioning. For a process that keeps going until a threshold is crossed, let $f(x)$ be the expected remaining number of steps given that x has already accumulated. Condition on the first step: it always costs 1 step, then the process continues from the new total. If the first step's size U is uniform on $[0, M]$, the expected continuation is the average of f over the reachable totals:

$$f(x) = 1 + \frac{1}{M} \int_x^{\text{threshold}} f(t)\, dt.$$

This is an integral equation for f; differentiating it (FTC) turns it into an ODE.

Problem 1 (BMT 2022/8)**.** At the Berkeley Mart for Technology, every item has a real-number cost independently and uniformly distributed from 0 to 2022. Sumith buys different items at the store until the total amount he spends strictly exceeds 1. Compute the expected value of the number of items Sumith buys.

Solution. Let $f(x)$ be the expected additional number of items bought given that x has already been spent; we want $f(0)$. Buying one item moves the total to $x + U$ with U uniform on $[0, 2022]$, and we continue only while the total stays ≤ 1, so first-step conditioning gives

$$f(x) = 1 + \frac{1}{2022} \int_x^1 f(t)\, dt.$$

Differentiating via FTC turns the integral equation into an ODE:

$$f'(x) = -\frac{f(x)}{2022}, \qquad \text{so} \qquad f(x) = Ce^{-x/2022}.$$

Since $f(1) = 1$ (one more item always suffices from exactly 1), $C = e^{1/2022}$, and $f(0) = e^{1/2022}$. Sanity check: each item alone exceeds \$1 with probability $\frac{2021}{2022}$, so an answer $e^{1/2022} \approx 1.0005$ barely above 1 is sensible. □

To match an integral to a proposed antiderivative, differentiate the answer and compare term by term.

Problem 2 (BMT 2021/5)**.** Suppose the following equality holds, where a, b, c are integers and K is the constant of integration:

$$\int \frac{\sin^a x - \cos^a x}{\sin^b x \cos^b x}\, dx = \frac{\csc^c x}{c} + \frac{\sec^c x}{c} + K.$$

If $a = 2021$, compute $a + b + c$.

Solution. Rewrite the integrand as $\frac{\sin^{a-b} x}{\cos^b x} - \frac{\cos^{a-b} x}{\sin^b x}$. Differentiating the proposed answer,

$$\frac{d}{dx}\left(\frac{\csc^c x}{c} + \frac{\sec^c x}{c}\right) = -\csc^c x \cot x + \sec^c x \tan x = -\frac{\cos x}{\sin^{c+1} x} + \frac{\sin x}{\cos^{c+1} x}.$$

Matching exponents term by term gives $a - b = 1$ and $b = c + 1$. With $a = 2021$, we get $b = 2020$ and $c = 2019$, so $a + b + c = 6060$. □

To maximize an integral over functions, complete the square in the integrand — the maximizer kills the square pointwise.

Problem 3 (BMT 2023/8)**.** Let $f : [1, \infty) \to \mathbb{R}$ be continuous such that

$$I(f) = \int_1^\infty \left(\sqrt{2023}\, x e^{-x} f(x) - \tfrac{1}{4} x^2 f(x)^2\right) dx$$

converges. If $I(f)$ is maximized over all continuous f, compute $f(1) + I(f)$.

Solution. Consider $-(axf(x) - be^{-x})^2 = 2abxe^{-x}f(x) - a^2x^2f(x)^2 - b^2e^{-2x}$. Matching coefficients with the integrand requires $2ab = \sqrt{2023}$ and $a^2 = \frac{1}{4}$, so $a = \frac{1}{2}$, $b = \sqrt{2023}$. Hence

$$I(f) = \int_1^\infty \left[2023e^{-2x} - \left(\tfrac{1}{2} x f(x) - \sqrt{2023}\, e^{-x}\right)^2\right] dx.$$

The squared term is nonpositive, so $I(f)$ is maximized exactly when it vanishes pointwise: $f(x) = \frac{2\sqrt{2023}\, e^{-x}}{x}$. Then $f(1) = \frac{2\sqrt{2023}}{e}$ and $I(f) = \int_1^\infty 2023e^{-2x}\, dx = \frac{2023}{2e^2}$. □

Before taking a limit at infinity, rescale coordinates so the geometry stabilizes.

Problem 4 (BMT 2025/7)**.** For positive real numbers a, let the circle $C(a)$ be the graph of $x^2 + (y - a)^2 = 4a$ in the coordinate plane. Let $f(a)$ be the fraction of the area of $C(a)$ that lies above the parabola $y = x^2$. Find $\lim_{a\to\infty} f(a)$.

Solution. Rescale with $x = 2\sqrt{a}\, u$ and $y = a + 2\sqrt{a}\, v$. The circle becomes the fixed unit circle $u^2 + v^2 = 1$, while the parabola becomes $a + 2\sqrt{a}\, v = 4au^2$. Dividing by $4a$, a point is above the parabola exactly when $u^2 \le \frac{1}{4} + \frac{v}{2\sqrt{a}}$. As $a \to \infty$ the last term vanishes (since $|v| \le 1$), so the parabola approaches the vertical segments $u = \pm\frac{1}{2}$ inside the circle, and the region above the parabola becomes the strip $|u| \le \frac{1}{2}$ of the unit disk. The lines $u = \pm\frac{1}{2}$ meet the unit circle at angles $\pm 60^\circ$ and $\pm 120^\circ$ (where $\cos\theta = \pm\frac{1}{2}$), so the strip splits into two 60° sectors, from 60° to 120° and from -120° to -60° (area $\frac{\pi}{6}$ each), and two triangles with a vertex at the center and base the chord $u = \pm\frac{1}{2}$ of length $\sqrt{3}$ (area $\frac{1}{2} \cdot \sqrt{3} \cdot \frac{1}{2} = \frac{\sqrt{3}}{4}$ each). Hence

$$\lim_{a\to\infty} f(a) = \frac{2 \cdot \frac{\pi}{6} + 2 \cdot \frac{\sqrt{3}}{4}}{\pi} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}.$$

□

At a tangency, the radius to the contact point is perpendicular to the curve — solve the orthogonality condition, then optimize.

Problem 5 (BMT 2018/8)**.** What is the maximum radius of a circle tangent to the curves $y = e^{-x^2}$ and $y = -e^{-x^2}$ at two points each?

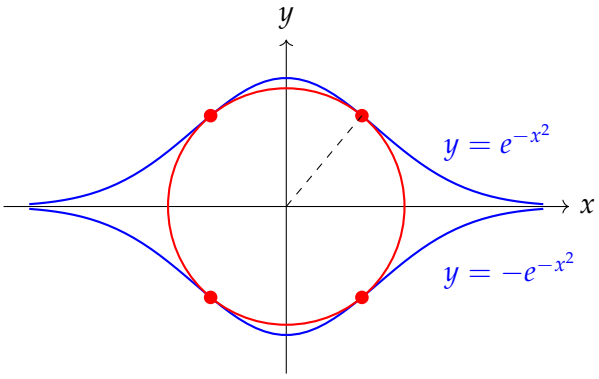
Solution. The picture is symmetric under $x \mapsto -x$ and $y \mapsto -y$, and we take the circle in the symmetric position the problem intends: between the two curves, centered at the origin. (The circle through $(0, \pm 1)$ touches each curve at just one point, so "two points each" means the contact points have $x \neq 0$.) Let (x, y) be a contact point on $y = e^{-x^2}$ with $x > 0$. The radial vector (x, y) must be perpendicular to the curve's tangent direction $(1, y') = (1, -2xy)$:

$$x \cdot 1 + y \cdot (-2xy) = x(1 - 2y^2) = 0,$$

so $y = \frac{1}{\sqrt{2}}$ (as $x \neq 0$). Then $e^{-x^2} = \frac{1}{\sqrt{2}}$ gives $x^2 = \frac{1}{2}\ln 2$, and

$$r^2 = x^2 + y^2 = \frac{\ln 2}{2} + \frac{1}{2} = \frac{1 + \ln 2}{2}, \qquad r = \sqrt{\frac{1 + \ln 2}{2}}.$$

Why this is the answer: the squared distance from the origin to the point (x, e^{-x^2}) is $D = x^2 + e^{-2x^2}$. With $t = x^2 \geq 0$, $D = t + e^{-2t}$ has $\frac{dD}{dt} = 1 - 2e^{-2t}$, negative for $t < \frac{1}{2}\ln 2$ and positive after, so D is minimized at $x^2 = \frac{1}{2}\ln 2$, exactly the contact point found above. Hence the circle of radius r is the largest circle about the origin that stays between the curves, touching each at the two closest points $x = \pm\sqrt{\frac{1}{2}\ln 2}$. Any other radius about the origin either misses the curves, touches only at $x = 0$, or crosses them, so this is the unique radius with two tangency points on each curve, and the maximum is $r = \sqrt{\frac{1+\ln 2}{2}}$. □



Answers: $e^{1/2022}$; 6060; $\frac{2\sqrt{2023}}{e} + \frac{2023}{2e^2}$; $\frac{1}{3} + \frac{\sqrt{3}}{2\pi}$; $\sqrt{\frac{1+\ln 2}{2}}$.