Basic Counting Techniques

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Most basic counting comes down to four moves. *Steps:* break the task into stages; if the *number* of options at each stage is the same no matter what happened before, multiply the option counts; an ordered list of k distinct objects from n gives $P(n,k) = n(n-1)\cdots(n-k+1)$. *Cases:* the outcomes split into disjoint possibilities — count each and add. *Complement:* the forbidden outcomes are easier to count — count everything, then subtract. *Groups:* objects that must stay together become one super-object; objects where order doesn't matter become a group, dividing out the $k!$ internal orders: $\binom{n}{k} = \frac{P(n,k)}{k!}$. Three examples below, then exercises.

Problem 1 (BMT 2018 Comb/1)**.** Bob has 3 different fountain pens and 11 different ink colors. How many ways can he fill his fountain pens with ink if he can only put one ink in each pen?

Solution. Fill the pens one at a time. There are 11 choices of ink for the first pen, 10 remaining choices for the second, and 9 for the third. Hence $11 \cdot 10 \cdot 9 = 990$ ways, i.e. $P(11,3)$. □

Alternative: Choose first, arrange second. Pick the 3 inks as a set: $\binom{11}{3}$ ways; then assign them to the 3 distinct pens: 3! orders. $\binom{11}{3} \cdot 3! = 165 \cdot 6 = 990$. (This is the Groups tool run in reverse: select ignoring order, then restore it.)

Problem 2 (BMT 2022/1)**.** For lunch, Lamy, Botan, Nene, and Polka each choose one of three options: a hot dog, a slice of pizza, or a hamburger. Lamy and Botan choose different items, and Nene and Polka choose the same item. In how many ways could they choose their items?

Solution. Nene and Polka always choose the same item, so bundle them into one super-chooser. Now there are three choosers: Lamy has 3 options, Botan must differ from Lamy (2 options), and the pair has 3 options. The total is $3 \cdot 2 \cdot 3 = 18$. □

Problem 3 (Classic)**.** How many distinct arrangements of the letters in MISSISSIPPI are there?

Solution. Imagine the 11 letters printed on distinct tiles: 11! arrangements. But the four I's are identical, so the 4! orders of the I-tiles all look the same — divide by 4!. Likewise divide by 4! for the S's and 2! for the P's. Hence $\dfrac{11!}{4!\,4!\,2!} = 34650$. □

Alternative: Place instead of dividing. Choose 4 of the 11 positions for the I's, 4 of the remaining 7 for the S's, 2 of the remaining 3 for the P's; M takes the last spot: $\binom{11}{4}\binom{7}{4}\binom{3}{2} = 330 \cdot 35 \cdot 3 = 34650$. Same count, built up rather than divided down.

Exercise 4 (BMT 2016/4)**.** Three 3-legged (distinguishable) Stanfurdians take off their socks and trade them with each other. How many ways is this possible if everyone ends up with exactly 3 socks and nobody gets any of their own socks? All socks originating from the Stanfurdians are distinguishable from each other. All Stanfurdian feet are indistinguishable from other feet of the same Stanfurdian. Hint: 1

Exercise 5 (BMT 2018 Comb/3)**.** Consider the 9×9 grid of lattice points $\{(x, y) \mid 0 \leq x, y \leq 8\}$. How many rectangles with nonzero area and sides parallel to the axes are there, with all corners on lattice points, such that the point $(4, 4)$ is not contained in the interior of the rectangle? ($(4, 4)$ may lie on the boundary.) Hint: 2

Exercise 6 (PUMaC 2010 Combinatorics #B3)**.** Let $\overline{xyz}$ represent the three-digit number with hundreds digit x, tens digit y, and units digit z, and similarly let $\overline{yz}$ represent the two-digit number with tens digit y and units digit z. How many three-digit numbers $\overline{abc}$, none of whose digits are 0, are there such that $\overline{ab} > \overline{bc} > \overline{ca}$? Hint: 3

Answers: 990; 18; 34650; 56; 1040; 120.

Hints

1. For each Stanfurdian, their 3 socks either all go to one other Stanfurdian or split 2–1. Case 1: some Stanfurdian sends all 3 to one Stanfurdian — show this forces a full 3-cycle (2 ways). Case 2: every Stanfurdian splits 2–1 — show all three splits must run the same way around the cycle, then count the $\binom{3}{2}$ choices per Stanfurdian.

2. Count all $\binom{9}{2}^2$ rectangles, then subtract those with $(4, 4)$ strictly inside. A rectangle contains $(4, 4)$ in its interior iff its two x-sides lie strictly on opposite sides of 4 — and the same for y.

3. Write $\overline{ab} > \overline{bc}$ as $10a + b > 10b + c$. If $a < b$, the left side is at most $10(b - 1) + 9$ — compare that with the right side. Once you have $a \geq b \geq c$, both two-digit inequalities become much simpler.