Divisibility: SFFT and Prime Exponents

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Two faces of the multiplicative structure of integers. **Simon's Favorite Factoring Trick:** clear denominators, complete the product, read off the finitely many divisor possibilities — or rewrite a fraction as a constant over a divisor plus an integer. **Prime exponents:** gcd takes the minimum exponent, lcm the maximum — so a gcd/lcm condition splits into independent one-prime caps, with $\gcd(x+a, x+b) = \gcd(x+a, b-a)$ killing x in one argument. Two examples below, then exercises.

Problem 1 (BMT 2019/4)**.** There exists one pair of positive integers a, b such that $100 > a > b > 0$ and $\frac{1}{a} + \frac{1}{b} = \frac{2}{35}$. Find $a + b$.

Solution.

$$\frac{1}{a} + \frac{1}{b} = \frac{2}{35} \iff 35(a+b) = 2ab \iff 4ab - 70a - 70b = 0.$$

Completing the product,

$$(2a - 35)(2b - 35) = 35^2 = 1225 = 5^2 \cdot 7^2.$$

From $a > b$ and $a < 100$: $2a - 35 > 2b - 35$ and $2a - 35 < 165$. Checking factor pairs of 1225:

$$\begin{aligned} (1225, 1),\ (245, 5),\ (175, 7) : a \geq 100 \quad & \text{out;} \\ (35, 35) : a = b \quad & \text{out;} \\ \text{both factors negative} : b \leq 0 \quad & \text{out;} \\ (49, 25) : a = 42,\ b = 30 \quad & \text{works.} \end{aligned}$$

Hence $a + b = 72$. □

Problem 2 (BMT 2023/6)**.** Let N be the number of positive integers x less than $210 \cdot 2023$ such that $\operatorname{lcm}(\gcd(x, 1734), \gcd(x+17, x+1732))$ divides 2023. Compute the sum of the prime factors of N with multiplicity.

Solution.

$$\gcd(x+17, x+1732) = \gcd(x+17, 1715), \qquad \operatorname{lcm}(u, v) \mid 2023 \iff u \mid 2023 \text{ and } v \mid 2023.$$

$$2023 = 7 \cdot 17^2, \qquad 1734 = 2 \cdot 3 \cdot 17^2, \qquad 1715 = 5 \cdot 7^3.$$

Prime by prime, $\gcd(x, 1734) \mid 2023$ and $\gcd(x+17, 1715) \mid 2023$ force

$$2 \nmid x, \quad 3 \nmid x, \qquad 5 \nmid x + 17, \quad 49 \nmid x + 17 \ (v_7 \leq 1),$$

with 17 unrestricted ($17^2 \parallel 1734$). By CRT these repeat mod $2 \cdot 3 \cdot 5 \cdot 49 = 1470$, giving

$$2 \cdot 1 \cdot 4 \cdot 48 = 384 = 2^7 \cdot 3$$

admissible residues per period (1 residue mod 2, 2 mod 3, 4 mod 5, 48 mod 49). Since $210 \cdot 2023 = 1470 \cdot 17^2$ and the endpoint $1470 \cdot 17^2$ is even (already excluded),

$$N = 384 \cdot 17^2 = 2^7 \cdot 3 \cdot 17^2.$$

Sum of prime factors with multiplicity:

$$7 \cdot 2 + 3 + 2 \cdot 17 = 51.$$

□

Exercise 3 (BMT 2020/7)**.** Compute the number of ordered triples of positive integers (a, b, c) such that $a + b + c + ab + bc + ca = abc + 1$. Hint: 1

Exercise 4 (BMT 2020/6)**.** Let N be the number of non-empty subsets T of $\{1, 2, \ldots, 2020\}$ satisfying $\max(T) > 1000$. Compute the largest integer k such that 3^k divides N. Hint: 2

Exercise 5 (CMIMC 2017 Number Theory #2)**.** Determine all possible values of $m + n$, where m and n are positive integers satisfying $\text{lcm}(m, n) - \gcd(m, n) = 103$. Hint: 3

Answers: 72; 51; 15; 2; 21, 105, 309.

Hints

1. Expand $(a-1)(b-1)(c-1) = abc - (ab+bc+ca) + (a+b+c) - 1$ and use the equation to replace $abc - (ab+bc+ca)$ by $a+b+c-1$: this gives $(a-1)(b-1)(c-1) = 2(a+b+c) - 2$. With $r = a-1$, $s = b-1$, $t = c-1$ this reads $rst = 2(r+s+t+2)$; bound the smallest of r, s, t, then factor what's left as in Simon's trick.

2. Count the complement and factor out a power of 2. Then reduce $2^{1020} - 1$ modulo 27 using $2^{18} \equiv 1 \pmod{27}$.

3. Write $m = d \cdot u$ and $n = d \cdot v$ where $d = \gcd(m, n)$, so $\text{lcm}(m, n) = d \cdot u \cdot v$ with u, v coprime. The equation becomes $d(uv - 1) = 103$ — and 103 is prime, so d has very few options.