Balls into Boxes: The Twelvefold Way

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Many counting problems are secretly about putting m balls into n boxes. The formula depends on three questions. *Are the balls distinct?* (Labeled balls can be told apart; unlabeled balls are identical.) *Are the boxes distinct?* (Labeled boxes: red vs. blue matters; unlabeled: only the pile sizes matter.) *How many balls can a box hold?* Any number, at most one, or at least one. The three answers pick one of twelve cells, each with its own formula — so *classify first, count second*. The tool names in the table are the techniques from the earlier handouts. Two worked examples below, then exercises.

	any per box	at most one per box	at least one per box
distinct balls, distinct boxes	n^m Steps	$P(n,m) = \frac{n!}{(n-m)!}$ Steps	$n! \cdot S(m,n)$ Inclusion–Exclusion
distinct balls, identical boxes	$\sum_{k \le n} S(m,k)$ Casework: no closed form	1 if $m \le n$, else 0	$S(m,n)$ Recursion
identical balls, distinct boxes	$\binom{m+n-1}{n-1}$ Stars and Bars	$\binom{n}{m}$ Groups	$\binom{m-1}{n-1}$ Stars and Bars
identical balls, identical boxes	partitions of m into $\le n$ parts Casework: no closed form	1 if $m \le n$, else 0	partitions of m into exactly n parts Casework

Here $S(m,n)$, the *Stirling number of the second kind*, counts the ways to split m distinct objects into n nonempty unlabeled groups; it satisfies the recursion $S(m,n) = n \cdot S(m-1,n) + S(m-1,n-1)$, the same shape as Pascal's rule. The cells with no closed form are still worth recognizing: the table's job there is to warn you off the wrong formula.

Problem 1 (BMT 2017/2)**.** Each BMT, every student chooses one of three focus rounds to take. Bob plans to attend BMT for the next 4 years and wants to figure out what focus round to take each year. Given that he wants to take each focus round at least once, how many ways can he choose which round to take each year?

Solution. Classify: the balls are the 4 years — distinct (a choice in year 1 is not a choice in year 2). The boxes are the 3 focus rounds — distinct. Each round taken at least once: the "at least one"

column. The table says inclusion–exclusion: count all 3^4 assignments, subtract those missing at least one round, add back those missing two:

$$3^4 - \binom{3}{1}2^4 + \binom{3}{2}1^4 = 81 - 48 + 3 = 36.$$

□

Problem 2 (BMT 2025/3)**.** In how many ways can ten identical white blocks be placed into a red box, a yellow box, a green box, and a blue box such that no box is empty, and at least two boxes contain the same number of blocks?

Solution. Classify: the 10 blocks are identical balls; the 4 colored boxes are distinct; no box empty is the "at least one" column. The table says stars and bars: pre-place one block in each box, then distribute the remaining 6 freely, giving $\binom{6+4-1}{4-1} = \binom{9}{3} = 84$ fillings. The extra condition needs a second move: count the complement, where all four counts are distinct. Distinct positive integers summing to 10 must be $1, 2, 3, 4$ (already the smallest possible sum), assignable to the 4 distinct boxes in $4! = 24$ ways. Hence $84 - 24 = 60$. □

Exercise 3. Six distinct books are placed onto four distinct shelves; a shelf may hold any number of books (possibly none). How many ways? Hint: 1

Exercise 4. Same six books and four shelves, but now each shelf must receive at least one book. How many ways? Hint: 2

Exercise 5 (SMT 2013 Team #3)**.** Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place? Hint: 3

Answers: 36; 60; 4096; 1560; 280.

Hints

1. Classify first: distinct balls, distinct boxes, any number per box. Each book independently chooses a shelf — which basic tool is that?
2. Classify: distinct balls, distinct boxes, at least one per box — the onto cell. Count all 4^6 placements, then alternately subtract and add back the placements missing one shelf, two shelves, and so on.
3. Classify first: the nine people are distinct balls and the three groups are unlabeled boxes. Count as if the groups were labeled (pick 3 for the first group, 3 for the second, 3 for the third), then divide by the number of ways to reorder the three groups.