Stars and Bars

David Liu

Splitting n identical objects among k distinct boxes is a distribution problem: only the count per box matters. Line up the n stars; choosing $k-1$ of the $n-1$ gaps for bars gives $\binom{n-1}{k-1}$ distributions with no box empty (allowing empty boxes, line up the n stars and the $k-1$ bars in a row and choose the bar positions: $\binom{n+k-1}{k-1}$). The art is spotting identical-vs.-distinct. *Bounds:* a box that must hold at least t? Give it t first, then distribute what's left with no restriction. Two examples below, then exercises.

Problem 1 (BMT 2025/3)**.** In how many ways can ten identical white blocks be placed into a red box, a yellow box, a green box, and a blue box such that no box is empty, and at least two boxes contain the same number of blocks?

Solution. The boxes are distinct but the blocks identical, so count distributions. With no box empty, stars and bars gives $\binom{10-1}{4-1} = \binom{9}{3} = 84$. The condition "at least two boxes match" is awkward directly, so count its complement: all four counts distinct. Positive integers summing to 10, pairwise distinct, must be $1+2+3+4$ (the smallest sum of four distinct positive integers is $1+2+3+4 = 10$, so it is the only way), arranged in $4! = 24$ ways. Hence $84 - 24 = 60$. □

Alternative: Skip stars and bars and case on the partition shape. Listing the ways to write 10 as a sum of four positive integers, only $1+2+3+4$ has all parts distinct. Summing the multinomial counts of the shapes with repeated parts gives

$$\underbrace{4}_{7111} + \underbrace{12}_{6211} + \underbrace{12}_{5311} + \underbrace{6}_{4411} + \underbrace{12}_{5221} + \underbrace{4}_{4222} + \underbrace{4}_{3331} + \underbrace{6}_{3322} = 60,$$

where a shape with two equal parts can be placed in $4!/2! = 12$ ways, with three equal parts in 4 ways, and with two pairs in $4!/(2!\,2!) = 6$ ways.

Problem 2 (BMT 2018 Comb/2)**.** At the Berkeley Math Tournament, teams are composed of 6 students, each of whom picks two distinct subject tests out of 5 choices. How many different distributions across subjects are possible for a team?

Solution. Write $(n_1, \ldots, n_5)$ for the number of students taking each subject: $n_i \geq 0$ and $\sum n_i = 12$. Stars and bars gives $\binom{12+5-1}{5-1} = \binom{16}{4} = 1820$ tuples. But each subject is taken by at most 6 students, so exclude the tuples with some $n_i \geq 7$. Fix i (5 choices), set $n_i' = n_i - 7 \geq 0$, and distribute the remaining 5 freely: $\binom{5+5-1}{4} = \binom{9}{4} = 126$ ways. Two subjects cannot both reach 7 ($7+7 > 12$), so the invalid tuples are disjoint: $5 \cdot 126 = 630$. Conversely, every tuple with all $n_i \leq 6$ really occurs: list the 12 subject slots in order (subject 1 written n_1 times, then subject 2 written n_2 times, and so on) and give student i slots i and $i+6$. These two slots hold different subjects, since a subject filling both would fill the 7 slots $i, \ldots, i+6$ in between. The answer is $1820 - 630 = 1190$. □

Exercise 3. How many distinct sums can be made from adding together exactly 8 numbers chosen from the set $\{1, 4, 7, 10\}$, where each number in the set is chosen at least once? Hint: 1

Exercise 4 (BMT 2023 Discrete Round #Tie 2)**.** Andrew, Benji, and Carlson want to split a pile of 5 indistinguishable left shoes and 7 indistinguishable right shoes. Andrew is practical and wants the same number of left and right shoes. Benji is greedy and wants the most shoes out of the three of them. Carlson is a trickster and wants Benji to have a different number of left and right shoes. How many ways are there to split up the shoes in a way that suits everyone's desires? Hint: 2

Exercise 5 (CMIMC 2016 Combinatorics #1)**.** The phrase "COLORFUL TARTAN" is spelled out with wooden blocks, where blocks of the same letter are indistinguishable. How many ways are there to distribute the blocks among two bags of different color such that neither bag contains more than one of the same letter? Hint: 3

Answers: 60; 1190; 13; 41; 16.

Hints

1. Force one of each: the problem becomes the achievable sums of 4 free picks. Stars and bars counts multisets, but different multisets can share a sum — what residue does every pick leave mod 3?
2. Andrew must take the same number k of left and right shoes — fix k, count the stars-and-bars distributions of the remaining left shoes and right shoes between Benji and Carlson, then use Benji's "most shoes" condition to bound what k can be. ("The most" means strictly more shoes than each of the other two; a tie does not count.)
3. Count the multiplicity of each letter. A letter appearing twice is forced: one block per bag. Only the singleton letters have a real choice — and the bags are distinct.