Inclusion–Exclusion

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Counting objects with "at least one of" several properties by adding the individual counts overcounts the overlaps. For two properties, $|A \cup B| = |A| + |B| - |A \cap B|$; for more, the signs alternate: add the singles, subtract the pairwise intersections, add back the triple intersections, and so on: each overcounted overlap gets added and subtracted until it is counted exactly once. The work is identifying what each intersection looks like. Two examples below, then exercises.

Problem 1 (BMT 2016/2)**.** How many integers from 1 to 2016 are divisible by 3 or 7, but not 21?

Solution. Let A be the multiples of 3 and B the multiples of 7. Then $|A| = \lfloor 2016/3 \rfloor = 672$, $|B| = \lfloor 2016/7 \rfloor = 288$, and $A \cap B$ is the multiples of 21, with $|A \cap B| = 96$. We want $(A \cup B) \setminus (A \cap B)$: first $|A \cup B| = 672 + 288 - 96 = 864$, then remove the intersection itself, giving $864 - 96 = 768$. □

Alternative: Make the sets disjoint first. Multiples of 3 that are not multiples of 21: $672 - 96 = 576$; multiples of 7 that are not multiples of 21: $288 - 96 = 192$. These groups don't overlap, so just add: $576 + 192 = 768$ — no alternating signs needed.

Problem 2 (BMT 2017/2)**.** Each BMT, every student chooses one of three focus rounds to take. Bob plans to attend BMT for the next 4 years and wants to figure out what focus round to take each year. Given that he wants to take each focus round at least once, how many ways can he choose which round to take each year?

Solution. There are $3^4 = 81$ assignments with no restriction. Let A_i be the assignments missing round i. Then $|A_i| = 2^4 = 16$, and $|A_i \cap A_j| = 1^4 = 1$ for $i \neq j$ (only one round ever taken); the triple intersection is empty. By inclusion–exclusion, the assignments missing at least one round number $\binom{3}{1} \cdot 16 - \binom{3}{2} \cdot 1 = 48 - 3 = 45$, so the assignments using every round are $81 - 45 = 36$. □

Alternative: Count the onto assignments directly. With 4 years and 3 rounds, some round is taken twice and the others once. Choose the doubled round (3 ways), then arrange the multiset $\{R, R, S, T\}$: $\frac{4!}{2!} = 12$ orders. Total $3 \cdot 12 = 36$.

Exercise 3. Bob attends BMT for 5 years instead of 4, still taking one of three focus rounds each year with each round taken at least once. How many ways? Hint: 1

Exercise 4 (PUMaC 2013 Combinatorics B #7)**.** You are eating at a fancy restaurant with a person you wish to impress. For some reason, you think that eating at least one spicy course and one meat-filled course will impress the person. The meal is five courses, with four options for each course. Each course has one option that is spicy and meat-filled, one option that is just spicy, one that is just meat-filled, and one that is neither spicy nor meat-filled. How many possible meals can you have? Hint: 2

Answers: 768; 36; 150; 961.

Hints

1. Same alternating sum as the second example, with 5 in place of 4: which rounds go missing?
2. Let A be the meals with no spicy course and B the meals with no meat-filled course; count A and B directly from the per-course options, then combine them with the alternating sum.