Recursion

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When the objects of size n are built out of smaller pieces, don't count directly: let a_n be the number you want, relate a_n to $a_{n-1}, a_{n-2}, \ldots$ by examining the first or last piece, and compute upward from small base cases. Two examples below, then exercises.

Problem 1 (BMT 2017/5)**.** How many subsets of $\{1, 2, \ldots, 9\}$ do not contain 2 adjacent numbers?

Solution. Let a_n count the valid subsets of $\{1, \ldots, n\}$. Look at whether n is in the subset: if not, there are a_{n-1} choices for the rest; if so, $n-1$ is excluded and there are a_{n-2} choices. Hence $a_n = a_{n-1} + a_{n-2}$, with $a_0 = 1$ and $a_1 = 2$. This is the Fibonacci recurrence: $3, 5, 8, 13, 21, 34, 55, 89$, so $a_9 = 89$. □

Alternative: Count by subset size instead of recursing. Choosing k non-adjacent numbers from $\{1, \ldots, 9\}$ means choosing k items with a forced gap between consecutive ones: delete one forced neighbor after each chosen number except the last, leaving k numbers chosen freely from $10-k$, i.e. $\binom{10-k}{k}$ ways. Summing $k = 0$ to 5 gives $1 + 9 + 28 + 35 + 15 + 1 = 89$.

Problem 2 (BMT 2021/7 (Challenge))**.** For a given positive integer n, at each step you may increase your number by one, or, if it is even, halve it. Let $\ell(n)$ be the minimum number of steps to reach 1 (e.g. $\ell(7) = 4$). How many positive integers n satisfy $\ell(n) \leq 12$?

Solution. First, the greedy strategy is optimal. For odd n there is no choice but to add 1. For even n, suppose an optimal sequence adds 1 a total of $k \geq 1$ times before its first halving (k is even, else the number stays odd and can't be halved). It reaches $(n+k)/2$ in $k+1$ steps — but halving first and then adding 1 a total of $k/2$ times reaches the same number in $1 + k/2 < k+1$ steps, a contradiction. So every optimal sequence halves each even number it meets.

Now count backwards by cost. Let a_m be the number of even n with $\ell(n) = m$ and b_m the number of odd n with $\ell(n) = m$. An even n of cost m halves to some n' of cost $m-1$, even or odd: $a_m = a_{m-1} + b_{m-1}$ (doubling inverts halving: $n \mapsto n/2$ pairs each even n of cost m with a unique n' of cost $m-1$). An odd $n \geq 3$ of cost m must go to the even $n+1$ of cost $m-1$; for $m \geq 3$ this is a bijection: every even $e \geq 4$ of cost $m-1$ comes from the odd $e-1$ of cost m, and the only even number missed is 2 itself (whose predecessor 1 has cost 0), so $b_m = a_{m-1}$. Thus $a_m = a_{m-1} + a_{m-2}$ for $m \geq 4$. With $a_0 = 0$, $a_1 = a_2 = 1$ and $b_0 = 1$, $b_1 = b_2 = 0$, the totals $a_m + b_m$ for $m = 0, \ldots, 12$ are $1, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$, which sum to 377. □

Exercise 3. Fourteen people sit at a circular table with fourteen distinct seats. They get up and sit back down so that each person sits in their original seat or an adjacent one, with no two sharing a seat. In how many ways can this occur? Hint: 1

Exercise 4 (PUMaC 2011 Combinatorics #A2 / #B3)**.** A set of n dominoes, each colored with one white square and one black square, is used to cover a $2 \times n$ board of squares. For $n = 6$, how many different patterns of colors can the board have? (For $n = 2$, this number is 6.) Hint: 2

Answers: 89; 377; 845; 328.

Hints

1. Fix person 1's new seat. If they stay put, the rest is a line of 13 chairs: let ℓ_m count the seatings for a line of m chairs and find its recurrence by looking at the first chair. Then handle person 1 moving to a neighboring chair: if person 2 follows person 1 to the right, what seat is left for person 3?

2. Let a_n count the valid color patterns of a $2 \times n$ board. Look at the two squares in the first column: if they differ, a vertical domino covers them and the rest is any valid pattern — but if they match, both horizontal dominoes are forced; what color must the second column be?