Structured Counting

David Liu

Some objects have rigid internal structure that the condition pins down almost completely. The move: prove the condition forces one specific structural shape, parametrize every valid object by that shape, and count the parameters. Below, functions whose arrows all flow into one point, and a number's carry pattern. Two examples, then exercises.

Problem 1 (Illustrative example)**.** Let $S = \{1,2,3,4\}$. How many functions $f : S \to S$ satisfy $f(f(f(s))) = 1$ for all $s \in S$? (Applying f three times is written f^3.)

Solution. Draw an arrow from each s to $f(s)$. Follow the arrows from 1: $1 \to f(1) \to f^2(1) \to f^3(1) = 1$, so 1 already lies on a directed cycle. If the cycle has length d, going around it returns to the start exactly after a multiple of d steps, so $f^3(1) = 1$ forces $d \mid 3$: $d = 1$ or 3. Length 3 is impossible: if $1 \to c_1 \to c_2 \to 1$, then $f^3(c_1) = c_1 \neq 1$, contradicting $f^3(c_1) = 1$. So $f(1) = 1$, and no other cycle can exist (its vertices would never reach 1).

Count by how many of $\{2,3,4\}$ map directly to 1. All three: 1 way. Exactly two: $\binom{3}{2}$ choices of the pair, and the third must map to one of the two: $3 \cdot 2 = 6$ ways. Exactly one: 3 choices for which vertex x; the other two must funnel into x without mapping to 1 themselves — both straight to x, or one through the other ($y \to x$ with $z \to y$, or $z \to x$ with $y \to z$): 3 ways, giving $3 \cdot 3 = 9$ ways. The total is $1 + 6 + 9 = 16$. □

Problem 2 (BMT 2023/3)**.** Find the number of positive integers n less than 10000 such that there are more 4's in the digits of $n + 1$ than in the digits of n.

Solution. Adding 1 only affects the trailing 9's (which become 0) and the digit just before them, say d, which becomes $d + 1$; every other digit is untouched. The trailing 9's contain no 4's, so the 4-count changes exactly at the carried digit: it increases by 1 if $d = 3$, decreases by 1 if $d = 4$, and is unchanged otherwise. Hence n must end in 3 followed by $j \geq 0$ nines. Counting by j (with $n < 10000$): $j = 0$ gives $1 + 9 + 90 + 900 = 1000$ numbers, $j = 1$ gives 100, $j = 2$ gives 10, and $j = 3$ gives 1, for a total of 1111. □

Exercise 3. Let $S = \{1,2,\ldots,6\}$. How many functions $f : S \to S$ satisfy $f^5(s) = 1$ for all $s \in S$? (f^5 means applying f five times.) This is a challenge exercise. Hint: 1

Exercise 4 (CMIMC 2016 Combinatorics Tiebreaker #1)**.** For a set $S \subseteq \mathbb{N}$, define $f(S) = \{\lceil \sqrt{s} \rceil \mid s \in S\}$. Find the number of sets T such that $|f(T)| = 2$ and $f(f(T)) = \{2\}$. Hint: 2

Answers: 16; 1111; 1296; 5043.

Hints

1. As in the first example: the arrows from 1 loop into a cycle whose length divides 5 — rule out the 5-cycle, so $f(1) = 1$ and no other cycle exists. Counting all the ways the remaining arrows flow into 1 needs one more idea beyond this handout.
2. Since $f(f(T)) = \{2\}$, every element t of $f(T)$ satisfies $\lceil \sqrt{t} \rceil = 2$ — list those t's. Then case on which two values $f(T)$ takes, and count the T's landing exactly on each.