Systematic Casework

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When configurations come in genuinely different shapes, don't force one formula: partition by a structural parameter — the number of blocks, the number of digits, a bounded statistic — so that each case is uniform and easy to count, then add. The skill is picking the parameter that makes every case clean. Two examples below, then exercises.

Problem 1 (BMT 2024/4)**.** Eight players are seated around a circular table. Each player is assigned to either Team Green or Team Yellow so that each team has at least one player. In how many ways can the players be assigned so that each player is on the same team as at least one adjacent player?

Solution. Read the seating clockwise as alternating blocks of Green and Yellow. There is an even number of blocks, and each block has size at least 2 (a block of size 1 would be an isolated player). With 8 players, either there are four blocks of size 2, or two blocks. Four blocks of 2: the pattern $GGYYGGYY$ has 4 distinct rotations ($GGYYGGYY$ repeats every 4 seats). Two blocks: the size splits are $(2,6)$, $(3,5)$, $(4,4)$, $(5,3)$, $(6,2)$, each with 8 rotations, giving $5 \cdot 8 = 40$. The total is $4 + 40 = 44$. □

Problem 2 (BMT 2021/5)**.** How many three-digit numbers abc have the property that $abc + cba$ is a palindrome? (Here cba is the digit-reversal of abc, and c may be 0.)

Solution. Write $abc + cba = 101(a + c) + 20b$ and case on the number of digits of the sum. If the sum has three digits, then $a + c \leq 9$. The units digit of $101(a + c) + 20b$ is always $a + c \pmod{10}$, fixed no matter what b is — so if $20b$ carried into the hundreds place, the hundreds digit would be $a + c + 1$ while the units digit stayed $a + c$, not a palindrome. Hence $2b < 10$, i.e. $b \leq 4$, giving the form $(a + c)(2b)(a + c)$. There are $9 + 8 + \cdots + 1 = 45$ pairs (a, c) with $a + c \leq 9$ and 5 choices for b, for $45 \cdot 5 = 225$ numbers. If the sum has four digits, it is at most 1998, so it is $1xy1$; then $a + c \equiv 1 \pmod{10}$, i.e. $a + c \in \{1, 11\}$. The value 1 gives a sum under 1000, so $a + c = 11$ and the sum is $1111 + 20b$, a palindrome only for $b = 0$. There are 8 pairs (a, c) with sum 11. The total is $225 + 8 = 233$. □

Exercise 3. Richard and Shreyas play 10 rounds of arm wrestling, one winner per round. Rounds won consecutively by the same person form a "streak". How many outcomes have strictly more than 3 streaks? Hint: 1

Exercise 4. Let $s(n)$ denote the sum of the digits of n. Compute the largest positive integer n such that $n = s(n)^2 + 2s(n) - 2$. Hint: 2, 3

Exercise 5 (PUMaC 2012 Combinatorics #B4)**.** For a set S of integers, define $\max(S)$ to be the maximal element of S. How many non-empty subsets $S \subseteq \{1, 2, \ldots, 10\}$ satisfy $\max(S) \leq |S| + 2$? Hint: 4

Answers: 44; 233; 932; 397; 175.

Hints

1. Count the complement: at most 3 streaks. For exactly k streaks, choose the $k-1$ cut positions among the 9 gaps between rounds, then pick the first streak's winner.

2. Bound the number of digits by growth: $d \geq 5$ gives $s(n) \leq 9d$, so $n = s(n)^2 + 2s(n) - 2 \leq 81d^2 + 18d - 2 < 10^{d-1}$, impossible (for $d = 5$: $n \leq 2113 < 10^4$; as d grows the left side grows far more slowly than 10^{d-1}). For $d = 4$: $s(n) \leq 36$ gives $n \leq 1366$; but then $s(n) \leq 21$ gives $n \leq 481 < 1000$, contradicting $d = 4$.

3. Now $s(n) \leq 27$. Mod 3, $n \equiv s(n)$ turns $n = s(n)^2 + 2s(n) - 2$ into $s(n) \equiv 1 \pmod 3$, leaving only a short list of possible $s(n)$ — and n grows with $s(n)$, so test the largest values first.

4. Case on the size $|S| = k$: the condition $\max(S) \leq k + 2$ pins down exactly which numbers S is allowed to use, and for a fixed k every case counts the same way. Add over k.