Basic Probability Tools

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Four moves that shrink a probability problem. Symmetry: when outcomes are interchangeable — heads $\leftrightarrow$ tails, the three ways to pair up four points — they have equal probabilities, so compute one and get the others free. Complement: when the event you want is messy to count directly, compute $1 - P(\text{the opposite})$ instead. Conditioning: throw away the worlds where a condition failed and renormalize, $P(A \mid B) = P(A \text{ and } B)/P(B)$; when a random quantity N governs the experiment, condition on its value and add over values. Bayes: you *observe a result* and want the probability of *which cause produced it* — list every way the result could have arisen, weighted by cause. Three examples below, then exercises.

Problem 1 (BMT 2020/2)**.** Haydn picks two different integers between 1 and 100, inclusive, uniformly at random. The probability that their product is divisible by 4 can be expressed in the form m/n, where m and n are relatively prime positive integers. Compute $m + n$.

Solution. Count the complement: the product is *not* divisible by 4. This happens iff both numbers are odd, or one is $\equiv 2 \pmod 4$ and the other is odd. There are $\binom{50}{2} = 1225$ odd–odd pairs and $25 \cdot 50 = 1250$ mixed pairs, out of $\binom{100}{2} = 4950$ total. Hence $P(\text{not divisible}) = 2475/4950 = 1/2$, so $P = 1/2$ and $m + n = 3$. □

Problem 2 (BMT 2018 Comb/7)**.** Let S be the set of line segments between any two vertices of a regular 21-gon. If we select two distinct line segments from S at random, what is the probability they intersect? (Segments sharing a vertex count as intersecting.)

Solution. There are $\binom{\binom{21}{2}}{2} = \binom{210}{2} = 21945$ pairs. Pairs sharing a vertex: $21\binom{20}{2} = 3990$, all intersecting. Pairs on four distinct vertices: $\binom{21}{4} \cdot 3 = 17955$. Group these by their four endpoints: each set of four points on the circle can be split into two segments in 3 ways, and exactly one of the three splits (the two diagonals of the convex quadrilateral) gives crossing segments, while the other two (opposite sides) do not. Since the three splits are equally likely (symmetry), exactly a third of these pairs cross: $17955/3 = 5985$ more intersecting pairs. Thus $P = (3990 + 5985)/21945 = 9975/21945 = 5/11$. □

Problem 3 (BMT 2019/2)**.** Sylvia has a bag of 10 coins. Nine are fair coins, but the tenth has tails on both sides. Sylvia draws a coin at random from the bag and flips it without looking. If the coin comes up tails, what is the probability that the coin she drew was the 2-tailed coin?

Solution. Count tail-faces: the 9 fair coins contribute 9 tail-faces and the double-tail coin contributes 2, for 11 tail-faces in total, all equally likely to be the observed tail. Two of them belong to the double-tail coin, so the probability is $2/11$ (this is Bayes in disguise: the 11 tail-faces are all the ways the observed tail could have arisen). □

Exercise 4. A fair coin is repeatedly flipped until 2019 consecutive flips are the same. Compute the probability that the first and last flips come up differently. Hint: 1

Exercise 5 (BMT 2023/5)**.** Kait rolls a fair 6-sided die until she rolls a 6. If she rolls a 6 on the Nth roll, she then rolls the die N more times. What is the probability that she rolls a 6 during these next N times? Hint: 2

Exercise 6 (PUMaC 2012 Combinatorics #A3 / #B5)**.** Jim has two fair 6-sided dice, one whose faces are labelled from 1 to 6, and the second whose faces are labelled from 3 to 8. Twice, he randomly picks one of the dice (each die equally likely) and rolls it. Given the sum of the resulting two rolls is 9, if $\frac{m}{n}$ is the probability he rolled the same die twice where m, n are relatively prime positive integers, then what is $m + n$? Hint: 3

Answers: 3; 5/11; 2/11; $\dfrac{2^{2018}-1}{2^{2019}-1}$; 6/11; 7.

Hints

1. Let x be the desired probability and condition on the first flip — say heads. Either flips 2 through 2019 are all heads, or the first tail appears somewhere; in the latter case symmetry restarts the game with heads and tails swapped.
2. Condition on $N = i$: the joint probability of "first 6 on roll i" and "no 6 in the next i rolls" is a geometric term. Sum over i and take the complement.
3. List all four ways the two rolls could have come about (AA, AB, BA, BB) and, for each, count the outcomes that give sum 9. The answer is the same-die ways divided by all the ways.