Linearity of Expectation

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$E[X+Y] = E[X] + E[Y]$ — always, with no independence needed. So to find an expected count, write it as a sum of 0/1 indicator variables: $E[\text{count}] = \sum P(\text{indicator} = 1)$, turning one hard expectation into many easy probabilities. Two examples below, then exercises.

Problem 1 (BMT 2020/4)**.** Three lights are placed horizontally in a line, all initially off. Every second, Neil picks one of the three lights uniformly at random and toggles it; toggling a light also toggles its immediate neighbors. The expected number of lights on after three seconds is m/n in lowest terms. Compute $m+n$.

Solution. By symmetry the two end lights behave identically. An end light toggles iff it or the middle light is picked (probability 2/3 per second), so it ends on iff it is toggled an odd number of times: $\binom{3}{1}(2/3)(1/3)^2 + \binom{3}{3}(2/3)^3 = 14/27$. The middle light toggles on every pick (directly or as a neighbor), so after three toggles it is certainly on. Hence $E = 2 \cdot \frac{14}{27} + 1 = \frac{55}{27}$, and $m+n = 82$. □

Problem 2 (BMT 2017/7)**.** A light is placed on every lattice point of the infinite 2D plane. Each light is on independently with probability $2^{-d/2}$, where $d = \max(|x|, |y|)$ is the Chebyshev distance from the origin. What is the expected number of lights whose four (Manhattan-)adjacent neighbors are all on?

Solution. By linearity, $E = \sum_p P(p\text{'s four neighbors all on})$. The origin contributes $2^{-4/2} = 1/4$. On ring $d \geq 1$, the 4 corners have neighbor distances $\{d, d, d+1, d+1\}$, contributing $4 \cdot 2^{-2d-1}$ each ring, and the $8d-4$ edge points have neighbor distances $\{d-1, d, d, d+1\}$, contributing $(8d-4)2^{-2d}$. Summing,

$$E = \frac{1}{4} + \sum_{d \geq 1} \left(4 \cdot 2^{-2d-1} + (8d-4)2^{-2d}\right) = \frac{1}{4} + \frac{2}{3} + \frac{20}{9} = \frac{113}{36}.$$

□

Exercise 3. Call the number of times the digits of a number change direction (increasing to decreasing or vice versa), ignoring equal consecutive digits, the *flux* of the number; e.g. the flux of 12333332 is 1. What is the average flux over the positive integers from 1 to 999? Hint: 1

Exercise 4 (PUMaC 2024 Combinatorics #A3 / #B5)**.** Joseph chooses a permutation of the numbers $1, 2, 3, 4, 5, 6$ uniformly at random. Then, he goes through his permutation, and deletes every number that is smaller than some number before it. For example, if he chooses the permutation $3, 2, 4, 5, 1, 6$, then he deletes 2 and 1, leaving him with $3, 4, 5, 6$. The expected number of numbers remaining can be expressed as m/n for relatively prime positive integers m and n. Find $m+n$. Hint: 2

Answers: 82; 113/36; 175/333; 69.

Hints

1. Write the flux as a sum of indicators — for a three-digit number there is only one place a direction change can occur. When is the middle digit a strict maximum or minimum? (Numbers below 100 have flux 0.)

2. Write the number of survivors as $X_1 + \cdots + X_6$, where $X_i = 1$ if position i survives. For position i to survive, its entry must be the largest among the first i entries — how likely is that?