Conditioning and Continuous Probability

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One idea, two settings. When a random process unfolds step by step, condition on the first step or the current state — in the first example below, conditioning on one digit gives the next digit's expected value. If the unknown keeps reappearing after conditioning, it satisfies an equation involving itself — write it down and solve. When the random choice is continuous — a point on a side, a number in an interval — condition on the values of uniform variables and integrate (or use area); if only *which region* each draw lands in matters, collapse the continuous law to those region probabilities and finish with discrete tools. Three examples below, then exercises.

Problem 1 (BMT 2021/8)**.** Consider the random real number $r = 0.p_0p_1p_2\ldots$, where p_0 is uniform on $\{0,\ldots,9\}$ and each p_{i+1} is uniform on $\{p_i, p_i+1,\ldots,9\}$. Compute $E[r]$.

Solution. By linearity, $E[r] = \sum_{i\geq 0} E[p_i]/10^{i+1}$. Conditioning on $p_i = d$ gives $E[p_{i+1} \mid p_i = d] = (d + (d+1) + \cdots + 9)/(10-d) = (9+d)/2$, so by total expectation $E[p_{i+1}] = 9/2 + E[p_i]/2$. With $E[p_0] = 9/2$ this solves to $E[p_i] = 9 - \frac{9}{2}\left(\frac{1}{2}\right)^i$. Hence

$$E[r] = \sum_{i\geq 0} \frac{9}{10^{i+1}} - \sum_{i\geq 0} \frac{9}{2}\frac{(1/2)^i}{10^{i+1}},$$

which are two geometric series summing to

$$\frac{9/10}{1-1/10} - \frac{9/20}{1-1/20} = 1 - \frac{9}{19} = \frac{10}{19}.$$

□

Problem 2 (BMT 2023/7)**.** Maria and Skyler split a 1-inch by 1-inch square cookie by joining two points chosen uniformly at random on distinct sides. Maria gets the larger piece. What is the expected excess of Maria's piece over Skyler's, in square inches?

Solution. The second point lands on an adjacent side with probability $2/3$ and the opposite side with probability $1/3$. On adjacent sides the smaller piece is a right triangle with legs $a, b \sim \text{Uniform}[0,1]$: $E[\text{area}] = E[ab]/2 = 1/8$, so the expected excess is $1 - 2\cdot\frac{1}{8} = \frac{3}{4}$. On opposite sides the pieces are trapezoids of areas $\frac{a+b}{2}$ and $1 - \frac{a+b}{2}$; with $S = a + b$ the expected excess is $E|1-S|$. Now S has the triangular density $f(s) = s$ on $[0,1]$ and $2-s$ on $[1,2]$, so $E|1-S| = \int_0^1 (1-s)s\,ds + \int_1^2 (s-1)(2-s)\,ds = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$. Total: $\frac{2}{3}\cdot\frac{3}{4} + \frac{1}{3}\cdot\frac{1}{3} = \frac{11}{18}$. □

Problem 3 (BMT 2022/7)**.** Luke flips cards from a shuffled 52-card deck until he has seen at least one ace and at least one king. What is the expected number of cards flipped?

Solution. Let A and K be the positions of the first ace and first king; we want $E[\max(A,K)] = E[A] + E[K] - E[\min(A,K)]$. The 4 aces split the other 48 cards into 5 equally-likely gaps, so $E[\text{cards before first ace}] = 48/5$ and $E[A] = 53/5$; likewise $E[K] = 53/5$. For the minimum, the 8 aces and kings split the remaining 44 cards into 9 gaps: $E[\min(A,K)] = 44/9 + 1 = 53/9$. Thus $E[\max(A,K)] = 2 \cdot \frac{53}{5} - \frac{53}{9} = \frac{689}{45}$. □

Exercise 4. Alice starts with an empty string and randomly appends one of the digits 2, 0, 1, 8 until the string ends in 2018. What is the probability she appends fewer than 9 digits before stopping? Hint: 1

Exercise 5 (BMT 2020/5)**.** Let P be the probability that the product of 2020 independent real numbers, each chosen uniformly at random from $[-1,2]$, is positive. Write $2P - 1 = (m/n)^b$ with $\gcd(m,n) = 1$ and b maximal. Compute $m + n + b$. Hint: 2

Exercise 6 (HMMT 2016 Theme #5)**.** Steph Curry is playing the following game and he wins if he has exactly 5 points at some time. Flip a fair coin. If heads, shoot a 3-point shot which is worth 3 points. If tails, shoot a free throw which is worth 1 point. He makes $\frac{1}{2}$ of his 3-point shots and all of his free throws. Find the probability he will win the game. (Note he keeps flipping the coin until he has exactly 5 or goes over 5 points.) Hint: 3

Answers: 10/19; 11/18; 689/45; 1279/65536; 2024; 140/243.

Hints

1. The pattern 2018 has no self-overlap: no proper prefix is also a suffix. If the string first ends in 2018 at position $n \geq 5$, it could not have ended at $n-1$, $n-2$, or $n-3$. Treat $n = 4, 5, 6, 7$ and $n = 8$ separately.

2. Only the sign of each factor matters: collapse the continuous law to $P(\text{negative}) = 1/3$. The product is positive iff an even number of factors are negative.

3. Let p_n be the probability of eventually hitting exactly 5 points starting from n points, and condition on the first coin flip and shot. A missed 3-pointer leaves the score at n, so p_n appears on both sides of its equation — solve for it, then work backward from $p_5 = 1$.