Number Theory 101

for the Crystal MATHCOUNTS team

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Warmup

1. Find the prime factorization of 2025 and 2026.
2. Is 91 prime? How about $987,654,321$?
3. How many different positive divisors does 7007 have?
4. What is the greatest common divisor of 1958 and 1960?

Take a minute to try these!

Divisibility Rules

- **2**: Last digit is even
- **3**: Sum of digits divisible by 3
- **4**: Last two digits divisible by 4
- **5**: Last digit is 0 or 5
- **6**: Divisible by both 2 and 3
- **8**: Last three digits divisible by 8
- **9**: Sum of digits divisible by 9
- **10**: Last digit is 0
- **11**: Alternating sum divisible by 11

For 11: Example: $7283 \to 7 - 2 + 8 - 3 = 10$

Instead of memorizing, try to understand why the rules work.

Example: Divisibility by 6

2018 Chapter Problem

Which digit should replace A in the units place of the number $987,654,32A$ so that the number is divisible by 6?

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Solution

To be divisible by 6, the number must be

- **divisible by 2**: A must be even (0, 2, 4, 6, 8).
- **divisible by 3**: sum of digits must be multiple of 3.
  digit sum: $9+8+7+6+5+4+3+2+A = 44+A$

Check even digits for A to see which one is divisible by 3:
$44+0=44$, $44+2=46$, $44+4=48$, $44+6=50$, $44+8=52$

Answer: 4

Time-Saving Insight

Smart Shortcut

When checking divisibility by 3 or 9, you can drop multiples of 3 from your sum!

For $987,654,32A$:

- 987: multiple of 3, can drop
- 654: multiple of 3, can drop
- We're left with $32A$: $3 + 2 + A = 5 + A$
- Need $5 + A$ divisible by 3, with A even
- $A = 4$ gives $5 + 4 = 9$

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Why does this work?

The sum of any 3 consecutive integers is always a multiple of 3.

Prime Factorization

Fundamental Theorem of Arithmetic

Every integer $n > 1$ has a **unique** prime factorization:

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$$

where p_i are distinct primes and a_i are positive integers.

Example

$2026 = 2 \times 1013$

$1001 = 7 \times 11 \times 13$

Number of Divisors

If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, Number of positive divisors of n

:

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$$

Why? For each prime factor p_i, we can choose exponent its $0, 1, 2, \ldots, a_i$, thus $a_i + 1$ choices.

Example (2025 State)

How many positive integer divisors does 432 have?

Solution

$432 = 2^4 \times 3^3$

$d(432) = (4 + 1) \times (3 + 1) = 5 \times 4 = \boxed{20}$

Challenge Problem

2013 State Problem

How many two-digit integers have exactly 8 positive factors?

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Solution

The prime factorization needs to be one of:

1. p^7: $2^7 = 128$ (too big) $\rightarrow$ 0 solutions
2. p^3q: When $p = 2$, $q = 3, 5, 7, 11$; when $p = 3$, $q = 2 \rightarrow 5$ solutions
3. pqr: $(2, 3, 5) = 30$, $(2, 3, 7) = 42$, $(2, 3, 11) = 66$, $(2, 3, 13) = 78$, $(2, 5, 7) = 70 \rightarrow 5$ solutions

Total: $0 + 5 + 5 = \boxed{10}$

Sum of Divisors

If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, Sum of all positive divisors of n:

$$\sigma(n) = (1 + p_1 + \cdots + p_1^{a_1}) \cdot (1 + p_2 + \cdots + p_2^{a_2}) \cdots$$

Example

Sum of divisors of $100 = 2^2 \cdot 5^2$:
$\sigma(100) = (1 + 2 + 4) \cdot (1 + 5 + 25) = 7 \times 31 = 217$

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Let S be the sum of all positive factors of $999,999$. What is the largest prime factor of S?

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Solution

$999,999 = 3^3 \times 7 \times 11 \times 13 \times 37$

$S = (1+3+9+27)(1+7)(1+11)(1+13)(1+37)$

Factors: 40, 8, 12, 14, 38

Largest prime factor among these is in $38 = 2 \times 19$

Answer: 19

Product of Divisors

If n has $d(n)$ divisors

Product of all positive divisors:

$$P(n) = n^{\frac{d(n)}{2}} = \sqrt{n^{d(n)}}$$

Why? Pair each divisor a with $\frac{n}{a}$.

Example (2011 National)

When all factors of 90 are multiplied together, the product is 90^k. Find k.

Solution

$90 = 2 \times 3^2 \times 5$

$d(90) = (1+1)(2+1)(1+1) = 12$

$k = \frac{d(90)}{2} = \boxed{6}$

GCD and LCM

GCD

Largest integer dividing both numbers

Example: $\gcd(12, 18) = 6$

LCM

Smallest integer divisible by both

Example: $\mathrm{lcm}(12, 18) = 36$

Using Prime Factorization

If $a = p_1^{x_1} p_2^{x_2} \cdots$ and $b = p_1^{y_1} p_2^{y_2} \cdots$:

$$\gcd(a, b) = p_1^{\min(x_1, y_1)} p_2^{\min(x_2, y_2)} \cdots$$

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Key Relationship

$$a \times b = \gcd(a, b) \times \text{lcm}(a, b)$$

Example: GCD with Factorials

2010 Chapter Problem

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$200,000 = 2^6 \times 5^5$.

Instead of doing a full prime factorization of $20!$, we only care about the power of 2 and 5:

- 2's: $16 = 2^4, 8 = 3^3$, so total number of 2's > 6.
- 5's: 5, 10, 15, 20 each gives a 5, so we 4 5's.

$\text{gcd} = 2^6 \times 5^4 = \boxed{40,000}$.

What is the power of p in $n!$ in general?

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What is the power of p in $n!$ in general?

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \left\lfloor \frac{n}{p^4} \right\rfloor + \dots$$

Final Challenge

Find all pairs of positive integers (x, y) such that:

$$\gcd(x, y) + \operatorname{lcm}(x, y) + xy = 50$$

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Hint

Let $d = \gcd(x, y)$, then $x = da$, $y = db$ where $\gcd(a, b) = 1$.
Then $\operatorname{lcm}(x, y) = dab$ and equation becomes:

$$d + dab + d^2ab = d(1 + ab + dab) = 50$$

Now we know d is a divisor of 50. In the worst case we can try all divisors and see which ones work!
You could also do SFFT (we will discuss that later).

Summary

Key Formulas

- $d(n) = \prod(a_i + 1)$
- $\sigma(n) = \prod \frac{p_i^{a_i+1}-1}{p_i-1}$
- $P(n) = n^{d(n)/2}$
- $a \times b = \text{gcd} \times \text{lcm}$

Strategies

- Know primes up to 100
- Use divisibility rules to help factorization

Practice makes perfect!