

MATHCOUNTS Number Theory 101

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Warmup

1. Find the prime factorization of 2025 and 2026.
2. Is 91 prime? How about 987,654,321?
3. How many different positive divisors does 7007 have?
4. What is the greatest common divisor of 1958 and 1960?

1 Divisibility Checks

Quick tests to determine if a number is divisible by certain integers:

- **2:** Last digit is even
- **3:** Sum of digits is divisible by 3
- **4:** Last two digits form a number divisible by 4
- **5:** Last digit is 0 or 5
- **6:** Divisible by both 2 and 3
- **8:** Last three digits form a number divisible by 8
- **9:** Sum of digits is divisible by 9
- **10:** Last digit is 0
- **11:** Alternating sum of digits is divisible by 11 (e.g., for 7283: $7 - 2 + 8 - 3 = 10$)

Example 1.1 (2018 Chapter)

Which digit should replace A in the units place of the number 987,654,32 A so that the number is divisible by 6?

Solution. To be divisible by 6, the number needs to be divisible by both 2 and 3. So the last digit is even, and all digits sum to a multiple of 3. Answer is 4. $\square$

Remark: When adding the digits, we can aggressively drop multiples of 3 in the process, because we only care about whether the sum is a multiple of 3. It is known that the sum of 3 consecutive integers is always a multiple of 3 (do you see why?), we don't need to worry about the 987 and 654 parts. So we just need 2A to be a multiple of 3. This kind of insight can save you time and is particularly useful in Sprint and Countdown rounds!

2 Prime Factorization

Make sure you can quickly decide if a number is prime for numbers up to 100.

2.1 Fundamental Theorem of Arithmetic

For any integer $n > 1$, there exists a unique representation:

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot \dots \cdot p_k^{a_k}$$

where $p_1, p_2, \dots, p_k$ are distinct primes and $a_1, a_2, \dots, a_k$ are positive integers.

Example 2.1

Find the prime factorization of 2026.

2.2 Number of Divisors

For $n = p_1^{a_1} \cdot p_2^{a_2} \cdot \dots \cdot p_k^{a_k}$, then the number of positive divisors of n is:

$$d(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$$

Do you see why? Hint: we do constructive counting and decide the exponents of each prime factor one by one, in k steps.

Example 2.2 (2025 State)

How many positive integer divisors does 432 have?

Solution. $432 = 2^4 \times 3^3$, so $d(432) = (4 + 1) \cdot (3 + 1) = \boxed{20}$.

□

Remark. What if we only want even divisors? How about divisors that are perfect squares?

Example 2.3 (2013 State)

How many two-digit integers have exactly 8 positive factors?

Solution. The number's prime factorization needs to be one of the following:

- p^7 . Note $2^7 = 128$, bigger than any two-digit integer, so this case has no solution.
- $p_1^3 p_2$. When $p_1 = 2$, p_2 can be one of 3, 5, 7, 11. When $p_1 = 3$, p_2 must be 2. $p_1 = 5$ would make p_1^3 exceed 100, so larger p_1 's won't work. There are 5 solutions in total for this case.
- $p_1 p_2 p_3$. Note the symmetry, so we let $p_1 < p_2 < p_3$. $\{(2, 3, 5), (2, 3, 7), (2, 3, 11), (2, 3, 13), (2, 5, 7)\}$ are the only possibilities.

Note that solutions in the different cases won't lead to the same number. So $\boxed{10}$ in total. $\square$

2.3 Sum of Divisors

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{a_1}) \cdot (1 + p_2 + p_2^2 + \cdots + p_2^{a_2}) \cdots$$

Example 2.4

Sum of all divisors of 100.

Solution. $\sigma(100) = (1 + 2 + 4) \cdot (1 + 5 + 25)$. If you expand the expression, you can find all divisors there in the form of $2^a 5^b$. $\square$

Example 2.5 (2025 State)

Let S be the sum of all the positive integers that are factors of 999,999. What is the largest prime number that is a factor of S ?

Solution. Prime factorize $999,999 = 9 \times 111 \times 1001 = 3^3 \times 7 \times 11 \times 13 \times 37$. So

$$S = (1 + 3 + 9 + 27) \cdot (1 + 7) \cdot (1 + 11) \cdot (1 + 13) \cdot (1 + 37).$$

Do not calculate S . Instead, note that $(1 + 37)$ has a prime factor $\boxed{19}$, which is bigger than others. $\square$

2.4 Product of Divisors

$$P(n) = n^{\frac{d(n)}{2}} = \sqrt{n^{d(n)}}$$

where $d(n)$ is the number of divisors.

Do you see why? Hint: pair up divisors a and $\frac{n}{a}$.

Example 2.6 (2011 National)

When all of the factors of 90 are multiplied together, the product can be expressed as 90^k . What is the value of k ?

Solution. $k = \frac{d(90)}{2} = \boxed{6}$. $\square$

Example 2.7

A positive integer n has 6 positive divisors. The product of all proper divisors (not including n itself) is 784. What is n ?

Solution. Note n is not in the product. Also note 1 is included but has no effect. So the other 4 factors form two pairs, each pair has a product of n , so $n^2 = 784$, and thus $n = 28$. $\square$

3 GCD and LCM

3.1 Definitions

- **GCD(a,b):** Largest positive integer dividing both a and b
- **LCM(a,b):** Smallest positive integer divisible by both a and b

3.2 Using Prime Factorization

If $a = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$ and $b = p_1^{y_1} p_2^{y_2} \cdots p_k^{y_k}$ (some exponents may be 0):

$$\begin{aligned}\text{GCD}(a, b) &= p_1^{\min(x_1, y_1)} p_2^{\min(x_2, y_2)} \cdots p_k^{\min(x_k, y_k)} \\ \text{LCM}(a, b) &= p_1^{\max(x_1, y_1)} p_2^{\max(x_2, y_2)} \cdots p_k^{\max(x_k, y_k)}\end{aligned}$$

Example 3.1 (2010 Chapter)

What is the greatest common factor of $20!$ and $200,000$?

Solution. It is easier to consider prime factorization of $200,000$. This is because $20!$ will contain many prime factors that are not factors of $200,000$.

$$200000 = 2 \times 10^5 = 2^6 \cdot 5^5.$$

We now need to find out the number of 2's and 5's in the prime factorization of $20!$. In fact, $16 = 2^4$ and $8 = 2^3$, so 16 and 8 alone gives more than 6 2's. Thus the number of 2's in the GCD should be 6.

For 5's, 5, 10, 15, 20 each contributes one 5 (and we don't have any number contributes more because $20 < 25$), so we have 4 5's in the GCD (note $4 < 5$).

$$\text{In sum we have } \text{GCD} = 2^6 \cdot 5^4 = \boxed{40000}.$$

$\square$

3.3 A Key Relationship

$$a \times b = \text{GCD}(a, b) \times \text{LCM}(a, b)$$

Example 3.2

The GCD of two numbers is 12 and their LCM is 72. If one of the numbers is 24, what is the other?

Solution. We have $24 \times b = 12 \times 72$, so $b = \frac{12 \times 72}{24} = 36$. □

Exercise 3.3. Find all pairs of positive integers x, y , such that $\text{GCD}(x, y) + \text{LCM}(x, y) + xy = 50$.